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Teaching and Learning Hyperbolic Functions (II): Other Trigonometric Properties and Their Inverses

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Abstract

As part of a larger project entitled "*Training and developing the competences of children, students and teachers to solve problems / exercises in Mathematics*", in a recent paper with the same generic name as this one and numbered with (I), I presented the definitions, the consequences immediate resulting from these and a series of 38 properties of hyperbolic functions, properties that we divided into four groups, as follows: A) "*Trigonometric*" properties - nine properties; B) The derivatives of hyperbolic functions - six properties; C) The primitives (indefinite integrals) of hyperbolic functions - six properties and D) The monotony and the invertibility of hyperbolic functions - 17 properties. In this paper we will continue this approach and will present and prove another 54 properties of these functions, properties that we will divide into three groups, as follows: E) Other properties "*trigonometric*" - 42 properties; F) Immediate properties of the inverse of hyperbolic functions - six properties and G) The derivatives of the inverse of hyperbolic functions - six properties. These properties, as well as others that we will present and prove later, will be used in various applications in Algebra or Mathematical Analysis.

Keywords: hyperbolic sine, hyperbolic cosine, hyperbolic tangent, hyperbolic cotangent, hyperbolic secant, hyperbolic cosecant.

1. Introduction

In (Vălcan, 2016) we started the presentation of a didactic exposition of fundamental properties of so-called "*hyperbolic functions*" in many aspects analogous to the usual trigonometric functions.

Also there I said that hyperbolic functions meet together many times in different physical and technical research, having a very important role in non-Euclidean geometry of Lobacevski participated in all relationships (interdependencies) this geometry. But independently of these annexes, the theory of hyperbolic functions can present a significant interest to a student or a teacher of Mathematics in secondary education because the analogy between the hyperbolic and trigonometric functions clarifies in a new face many problems of trigonometry. As I said hyperbolic functions occur naturally as simple combinations of exponential function, e^x, a function that is much studied in School Mathematics. Indeed, the two main functions, hyperbolic cosine, and hyperbolic sine is semisum or semi difference of e^x and e^{-x}, see the following equalities (2.1) and (2.2). We have to admit that in undergraduate education in Romania, these functions are almost unknown, so students and teachers, despite the fact that they present many similarities to the trigonometric functions and, in addition, have numerous applications in integral calculus. I must admit that no students from the Faculty of Mathematics do not really know these things. In conclusion, we can say that literature the domain in Romania is very poor in providing information about the hyperbolic functions and their applications. Not even abroad things are not so good – see (Vălcan, 2016, Introduction). In this second paper, continuing with the ideas from (Vălcan, 2016) we will present another 54 properties of these functions.

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To the reader interested in these issues we also recommend reading the bibliographic sources (Abramowitz & Stegun, 1973), (Anderson, 1999), (Beyer, 1987), (Harris & Stocker, 1998), (Jeffrey, 2000), (Yates, 1953), (Zwillinger, 1995) and but also other bibliographic sources that are available online. To give coherence and consistency to the ideas of these two works, we will recall the results of the first work; that and because we will always refer to them.

2. The definitions of hyperbolic functions

In this first section we have defined the hyperbolic functions and we have presented the first their properties.

Definition 2.1: *The function sh :* $\boldsymbol{R} \to \boldsymbol{R}$*, given by law, for every* $x\in\boldsymbol{R}$,

$$sh(x)=\frac{e^x-e^{-x}}{2}, \tag{2.1}$$

is called ***hyperbolic sine (in latin, sinus hyperbolus)*** *by argument x.*

Definition 2.2: *The function ch :* $\boldsymbol{R} \to [1,+\infty)$*, given by law, for every* $x\in\boldsymbol{R}$,

$$ch(x)=\frac{e^x+e^{-x}}{2}, \tag{2.2}$$

is called ***hyperbolic cosine (in latin, cosinus hyperbolus)*** *by argument x.*

Definition 2.3: *The function th :* $\boldsymbol{R} \to (-1,1)$*, given by law, for every* $x\in\boldsymbol{R}$,

$$th(x)=\frac{sh(x)}{ch(x)}, \tag{2.3}$$

is called ***hyperbolic tangent*** *by argument x.*

Definition 2.4: *The function cth :* $\boldsymbol{R}^* \to (-\infty,-1)\cup(1,+\infty)$*, given by law, for every* $x\in\boldsymbol{R}^*$,

$$cth(x)=\frac{ch(x)}{sh(x)}, \tag{2.4}$$

is called ***hyperbolic cotangent*** *by argument x.*

Definition 2.5: *The function sch :* $\boldsymbol{R} \to (0,1]$*, given by law, for every* $x\in\boldsymbol{R}$,

$$sch(x)=\frac{1}{ch(x)}, \tag{2.5}$$

is called ***hyperbolic secant*** *by argument x.*

Definition 2.6: *The function csh :* $\boldsymbol{R}^* \to \boldsymbol{R}^*$*, given by law, for every* $x\in\boldsymbol{R}^*$,

$$csh(x)=\frac{1}{sh(x)}, \tag{2.6}$$

is called ***hyperbolic cosecant*** *by argument x.*

Remarks 2.7: *From the above definitions it follows that:*

1) *The functions sh and ch are linear combinations of exponential functions:*

$x \mapsto e^x$, *respectively* $x \mapsto e^{-x}$, $x\in\boldsymbol{R}$;

and vice versa, that is, the functions e^x *and* e^{-x} *are linear combinations of the functions:*

$x \mapsto sh(x)$, *respectively* $x \mapsto ch(x)$, $x\in\boldsymbol{R}$;

because, for every $x\in\boldsymbol{R}$:

$$e^x=ch(x)+sh(x) \tag{2.7}$$

and

$$e^{-x}=ch(x)-sh(x). \tag{2.8}$$

2) *For every* $x\in\boldsymbol{R}$,

$$sh(x)=\frac{e^{2x}-1}{2e^x}=\frac{1-e^{-2x}}{2e^{-x}}. \tag{2.1′}$$

3) *For every* $x\in\boldsymbol{R}$,

$$ch(x)=\frac{e^{2x}+1}{2e^x}=\frac{1+e^{-2x}}{2e^{-x}}. \tag{2.2′}$$

4) *For every* $x\in\boldsymbol{R}$,

$$th(x)=\frac{e^x-e^{-x}}{e^x+e^{-x}}=\frac{e^{2x}-1}{e^{2x}+1}=\frac{1-e^{-2x}}{1+e^{-2x}}. \tag{2.3′}$$

5) *For every* $x\in\boldsymbol{R}^*$,

$$cth(x)=\frac{e^x+e^{-x}}{e^x-e^{-x}}=\frac{e^{2x}+1}{e^{2x}-1}=\frac{1+e^{-2x}}{1-e^{-2x}}. \tag{2.4′}$$

6) *For every* $x\in\boldsymbol{R}$,

$$sch(x)=\frac{2}{e^x+e^{-x}}=\frac{2e^x}{e^{2x}+1}=\frac{2e^{-x}}{1+e^{-2x}}. \tag{2.5′}$$

5) *For every* $x\in\boldsymbol{R}^*$,

$$csh(x)=\frac{2}{e^x-e^{-x}}=\frac{2e^x}{e^{2x}-1}=\frac{2e^{-x}}{1-e^{-2x}}. \tag{2.6′}$$

6) *To make analogies with the trigonometric functions, but for abbreviations, we use the following notations further*:

a) *for every* $x\in\boldsymbol{R}$,

$sh(x)\overset{not.}{=}shx$, $ch(x)\overset{not.}{=}chx$, $th(x)\overset{not.}{=}thx$, $sch(x)\overset{not.}{=}schx$,

b) *for every* $x\in\boldsymbol{R}^*$,

$cth(x)\overset{not.}{=}cthx$, $csh(x)\overset{not.}{=}cshx$;

but any expression including simple fraction, which will be the argument of one of these functions will be put between brackets. Also here we specify that in some papers functions: sh, ch, th, cth, sch, csh, they are denoted, respectively: sinh, cosh, tanh, cotanh, sech, cosech. □

Remarks 2.8: *From the above definitions and remarks it follows that*:

1) *The function sh is odd, i.e.: for every* $x\in\boldsymbol{R}$,

$sh(-x)=-shx.$ (2.9)

2) *The function ch is even, i.e.: for every* $x\in\boldsymbol{R}$,

$ch(-x)=chx.$ (2.10)

3) *The function th is odd, i.e.: for every* $x\in\boldsymbol{R}$,

$th(-x)=-thx.$ (2.11)

4) *The function cth is odd, i.e.: for every* $x\in\boldsymbol{R}^*$,

$cth(-x)=-cthx.$ (2.12)

5) *The function sch is even, i.e.: for every* $x\in\boldsymbol{R}$,

$sch(-x)=schx.$ (2.13)

6) *The function csh is odd, i.e.: for every* $x\in\boldsymbol{R}^*$,

$csh(-x)=-cshx.$ (2.14)

7) *For every* $x\in\boldsymbol{R}$,

$chx\geq 1$; (2.15)

8) *For every orice* $x\in\boldsymbol{R}$,

$thx\in(-1,1).$ (2.16)

3. Fundamental properties of hyperbolic functions

In this section of the paper (Vălcan, 2017) we have presented the fundamental properties of hyperbolic functions; we referred here to the first 38 of these properties, divided into four groups:

A. *„Trigonometric" properties – nine properties*;

B. *The derivatives of hyperbolic functions – six properties*;

C. *The primitives (indefinite integrals) of hyperbolic functions – six properties*;

D. *The monotony and the invertibility of hyperbolic functions – 17 properties.*

All of these properties have been prooved at least in a way.

Proposition 3.1: *The following statements hold*:

A. „Trigonometric" properties

1) *For every* $x\in\boldsymbol{R}$,

$ch^2x-sh^2x=1.$ (3.1)

2) *For every* $x, y\in\boldsymbol{R}$,

$sh(x+y)=shx\cdot chy+chx\cdot shy.$ (3.2)

3) *For every* $x, y\in\boldsymbol{R}$,

$sh(x-y)=shx\cdot chy-chx\cdot shy.$ (3.3)

4) *For every* $x, y\in\boldsymbol{R}$,

$ch(x+y)=chx\cdot chy+shx\cdot shy.$ (3.4)

5) *For every* $x, y\in\boldsymbol{R}$,

$ch(x-y)=chx\cdot chy-shx\cdot shy.$ (3.5)

6) *For every* $x, y\in\boldsymbol{R}$,

$$th(x+y)=\frac{thx+thy}{1+thx\cdot thy}. \tag{3.6}$$

7) *For every x, y∈**R**,*

$$th(x-y)=\frac{thx-thy}{1-thx\cdot thy}. \tag{3.7}$$

8) *For every x, y∈**R***, with the property that x+y≠0,*

$$cth(x+y)=\frac{cthx\cdot cthy+1}{cthx+cthy}. \tag{3.8}$$

9) *For every x, y∈**R***, with the property that x≠y,*

$$cth(x-y)=\frac{cthx\cdot cthy-1}{cthy-cthx}. \tag{3.9}$$

B. The derivatives of hyperbolic functions

10) *For every x∈**R**,*

$$(shx)'=chx. \tag{3.10}$$

11) *For every x∈**R**,*

$$(chx)'=shx. \tag{3.11}$$

12) *For every x∈**R**,*

$$(thx)'=\frac{1}{ch^2x}=sch^2x. \tag{3.12}$$

13) *For every x∈**R***,*

$$(cthx)'=-\frac{1}{sh^2x}=-csh^2x. \tag{3.13}$$

14) *For every x∈**R**,*

$$(schx)'=-\frac{shx}{ch^2x}=-thx\cdot schx. \tag{3.14}$$

15) *For every x∈**R***,*

$$(cshx)'=-\frac{chx}{sh^2x}=-cthx\cdot cshx. \tag{3.15}$$

C. The primitives (indefinite integrals) of hyperbolic functions

16) *For every x∈**R**,*

$$\int shx\cdot dx=chx+C. \tag{3.16}$$

17) *For every x∈**R**,*

$$\int chx\cdot dx=shx+C. \tag{3.17}$$

18) *For every x∈**R**,*

$$\int thx\cdot dx=ln(chx)+C. \tag{3.18}$$

$$\int cthx\cdot dx=ln|shx|+C. \tag{3.19}$$

20) *For every x∈**R**,*

$$\int schx\cdot dx=arctg(shx)+C. \tag{3.20}$$

21) *For every x∈**R***,*

$$\int cshx\cdot dx=\frac{1}{2}\cdot ln\left(\frac{chx-1}{chx+1}\right)+C. \tag{3.21}$$

19) *For every x∈**R***,*

D. The monotony and the invertibility of hyperbolic functions

22) *The function sh is strictly increasing on **R**.*

23) *The function sh is invertible and its inverse is the function:*

$sh^{-1}:\mathbf{R}\to\mathbf{R}$,

*where, for every x∈**R**,*

$$sh^{-1}(x)=ln(x+\sqrt{x^2+1}). \tag{3.22}$$

24) *The function ch is strictly decreasing on (-∞,0) and strictly increasing on (0,+∞).*

25) *The function ch_1 - the restriction of function ch to the interval (-∞,0], is invertible and its inverse is the function:*

$ch_1^{-1}:[1,+\infty)\to(-\infty,0)$,

where, for every x∈[1,+∞),

$$ch_1^{-1}(x)=ln(x-\sqrt{x^2-1}). \tag{3.23}$$

26) *The function ch_2 - the restriction of function ch to the interval [0,+∞), is invertible and its inverse is the function:*

$ch_2^{-1}:[1,+\infty)\to[0,+\infty)$,

where, for every x∈[1,+∞),

$$ch_2^{-1}(x)=ln(x+\sqrt{x^2-1}). \tag{3.24}$$

27) *The function th is strictly increasing on **R**.*

28) *The function th is invertible and its inverse is the function:*

$th^{-1}:(-1,1)\to\mathbf{R}$,

where, for every x∈(-1,1),

$$th^{-1}(x)=\frac{1}{2}\cdot ln\left(\frac{1+x}{1-x}\right). \tag{3.25}$$

29) *The function cth is strictly decreasing both on $(-\infty,0)$, as well as on $(0,+\infty)$.*

30) *The function cth_1 - the restriction of function cth to the interval $(-\infty,0)$, is invertible and its inverse is the function:*

$cth_1^{-1} : (-\infty,-1) \to (-\infty,0)$,

where, for every $x\in(-\infty,-1)$,

$$cth_1^{-1}(x)=\frac{1}{2}\cdot ln\left(\frac{x+1}{x-1}\right). \tag{3.26}$$

31) *The function cth_2 - the restriction of function cth to the interval $(0,+\infty)$, is invertible and its inverse is the function:*

$cth_2^{-1} : (1,+\infty) \to (0,+\infty)$,

where, for every $x\in(1,+\infty)$,

$$cth_2^{-1}(x)=\frac{1}{2}\cdot ln\left(\frac{x+1}{x-1}\right). \tag{3.26'}$$

32) *The function cth is invertible and its inverse is the function:*

$cth^{-1} : (-\infty,-1)\cup(1,+\infty) \to \boldsymbol{R}^*$,

where, for every $x\in(-\infty,-1)\cup(1,+\infty)$,

$$cth^{-1}(x)=\frac{1}{2}\cdot ln\left(\frac{x+1}{x-1}\right). \tag{3.26''}$$

33) *The function sch is strictly increasing on $(-\infty,0]$ and strictly decreasing on $[0,+\infty)$.*

34) *The function sch_1 - the restriction of function sch to the interval $(-\infty,0]$, is invertible and its inverse is the function:*

$sch_1^{-1} : (0,1] \to (-\infty,0]$,

where, for every $x\in(0,1]$,

$$sch_1^{-1}(x)=ln\left(\frac{1-\sqrt{1-x^2}}{x}\right). \tag{3.27}$$

35) *The function sch_2 - the restriction of function sch to the interval $[0,+\infty)$, is invertible and its inverse is the function:*

$sch_2^{-1} : (0,1] \to [0,+\infty)$,

where, for every $x\in(0,1]$,

$$sch_2^{-1}(x)=ln\left(\frac{1+\sqrt{1-x^2}}{x}\right). \tag{3.28}$$

36) *The function csh is strictly decreasing both on $(-\infty,0)$, as well as on $(0,+\infty)$.*

37) *The function csh_1 - the restriction of function csh to the interval $(-\infty,0)$ is invertible and its inverse is the function:*

$csh_1^{-1} : (-\infty,0) \to (-\infty,0)$,

where, for every $x\in(-\infty,0)$,

$$csh_1^{-1}(x)=ln\left(\frac{1-\sqrt{1+x^2}}{x}\right). \tag{3.29}$$

38) *The function csh_2 - the restriction of function csh to the interval $(0,+\infty)$, is invertible and its inverse is the function:*

$csh_2^{-1} : (0,+\infty) \to (0,+\infty)$,

where, for every $x\in(0,+\infty)$,

$$csh_2^{-1}(x)=ln\left(\frac{1+\sqrt{1+x^2}}{x}\right). \tag{3.30}$$

4. Other trigonometric properties and their inverses

First, before exposing new results, it is necessary here to make some remarks, analogous to those of Remarks 2.7, point 6):

Remark 4.1: *To make analogies with the inverse of trigonometric functions, but for abbreviations, we use the following notations further*:

a) *for every* $x\in\mathbf{R}$,

$sh^{-1}(x) \overset{not.}{=} sh^{-1}x$;

b) *for every* $x\in[1,+\infty)$,

$ch_1^{-1}(x) \overset{not.}{=} ch_1^{-1}x$ *and* $ch_2^{-1}(x) \overset{not.}{=} ch_2^{-1}x$;

c) *for every* $x\in(-1,1)$,

$th^{-1}(x) \overset{not.}{=} th^{-1}x$;

d) *for every* $x\in(-\infty,-1)$,

$cth_1^{-1}(x) \overset{not.}{=} cth_1^{-1}x$;

e) *for every* $x\in(1,+\infty)$,

$cth_2^{-1}(x) \overset{not.}{=} cth_2^{-1}x$;

f) *for every* $x\in(-\infty,-1)\cup(1,+\infty)$,

$cth^{-1}(x) \overset{not.}{=} cth^{-1}x$;

g) *for every* $x\in(0,1]$,

$sch_1^{-1}(x) \overset{not.}{=} sch_1^{-1}x$ *and* $sch_2^{-1}(x) \overset{not.}{=} sch_2^{-1}x$;

h) *for every* $x\in(-\infty,0)$,

$csh_1^{-1}(x) \overset{not.}{=} csh_1^{-1}x$;

i) *for every* $x\in(0,+\infty)$,

$csh_2^{-1}(x) \overset{not.}{=} csh_2^{-1}x$;

but any expression including simple fraction, which will be the argument of one of these functions, will be put between brackets. Here we mention that in some papers *the functions:* sh^{-1}, ch_1^{-1}, ch_2^{-1}, th^{-1}, cth_1^{-1}, cth_2^{-1}, cth^{-1}, sch_1^{-1}, sch_2^{-1}, csh_1^{-1} *and* csh_2^{-1}, they are denoted respectively by*: arcsinh, arccosh, arctanh, arccotanh, arcsech and arccosech.* □

Proposition 4.2: *The following statements hold*:

E. Other properties „trigonometric"

1) *For every* $x\in\mathbf{R}$,

$$shx=2\cdot sh\left(\frac{x}{2}\right)\cdot ch\left(\frac{x}{2}\right). \tag{4.1}$$

2) *For every* $x\in\mathbf{R}$,

$$sh(2x)=2\cdot shx\cdot chx. \tag{4.2}$$

3) *For every* $x\in\mathbf{R}$,

$$chx=sh^2\left(\frac{x}{2}\right)+ch^2\left(\frac{x}{2}\right). \tag{4.3}$$

4) *For every* $x\in\mathbf{R}$,

$$ch(2x)=sh^2x+ch^2x; \tag{4.4}$$

$$ch(2x)=2\cdot ch^2x-1; \tag{4.4′}$$

$$ch(2x)=2\cdot sh^2x+1. \tag{4.4″}$$

5) *For every* $x\in(-\infty,0)$,

$$sh\left(\frac{x}{2}\right)=-\sqrt{\frac{chx-1}{2}} \tag{4.5}$$

and for every $x\in[0,+\infty)$,

$$sh\left(\frac{x}{2}\right)=\sqrt{\frac{chx-1}{2}}. \tag{4.5′}$$

6) *For every* $x\in\mathbf{R}$,

$$ch\left(\frac{x}{2}\right)=\sqrt{\frac{chx+1}{2}}. \tag{4.6}$$

7) *For every* $x\in\mathbf{R}$,

$$thx=\frac{2\cdot th\left(\frac{x}{2}\right)}{1+th^2\left(\frac{x}{2}\right)}. \tag{4.7}$$

8) *For every* $x\in\mathbf{R}$,

$$th(2x)=\frac{2\cdot thx}{1+th^2x}. \tag{4.8}$$

9) *For every* $x\in\mathbf{R}$,

$$th\left(\frac{x}{2}\right)=\begin{cases}-\sqrt{\dfrac{chx-1}{chx+1}}, & if\ \ x<0\\ \sqrt{\dfrac{chx-1}{chx+1}}, & if\ \ x\geq 0\end{cases}=\frac{shx}{chx+1}=\frac{chx-1}{shx}=cthx-cshx. \tag{4.9}$$

10) *For every* $x\in\mathbf{R}^*$,

$$cthx=\frac{cth^2\left(\frac{x}{2}\right)+1}{2\cdot cth\left(\frac{x}{2}\right)}. \tag{4.10}$$

$$cth(2x)=\frac{cth^2x+1}{2\cdot cthx}. \tag{4.11}$$

11) *For every* $x\in\mathbf{R}^*$,

12) *For every* $x\in\boldsymbol{R}^*$,

$$cth\left(\frac{x}{2}\right)=\begin{cases}-\sqrt{\dfrac{chx+1}{chx-1}}, & if\ \ x<0\\ \sqrt{\dfrac{chx+1}{chx-1}}, & if\ \ x\geq 0\end{cases}=\frac{chx+1}{shx}=\frac{shx}{chx-1}=cthx+cshx. \qquad (4.12)$$

13) *For every* $x, y, z\in\boldsymbol{R}$,

$sh(x+y+z)=shx\cdot chy\cdot chz+chx\cdot shy\cdot chz+chx\cdot chy\cdot shz+shx\cdot shy\cdot shz.$ (4.13)

14) *For every* $x\in\boldsymbol{R}$,

$sh(3x)=shx\cdot(4\cdot sh^2x+3)=shx\cdot(4\cdot ch^2x-1).$ (4.14)

15) *For every* $x, y, z\in\boldsymbol{R}$,

$ch(x+y+z)=chx\cdot chy\cdot chz+shx\cdot shy\cdot chz+shx\cdot chy\cdot shz+chx\cdot shy\cdot shz.$ (4.15)

16) *For every* $x\in\boldsymbol{R}$,

$ch(3x)=chx\cdot(4\cdot ch^2x-3)=chx\cdot(4\cdot sh^2x+1).$ (4.16)

17) *For every* $x, y, z\in\boldsymbol{R}$,

$$th(x+y+z)=\frac{thx+thy+thz+thx\cdot thy\cdot thz}{1+thx\cdot thy+thy\cdot thz+thx\cdot thz}. \qquad (4.17)$$

18) *For every* $x\in\boldsymbol{R}$,

$$th(3x)=\frac{3\cdot thx+th^3x}{1+3\cdot th^2x}. \qquad (4.18)$$

19) *For every* $x, y, z\in\boldsymbol{R}^*$, *such that* $x+y, y+z, x+z, x+y+z\in\boldsymbol{R}^*$,

$$cth(x+y+z)=\frac{cthx\cdot cthy\cdot cthz+cthx+cthy+cthz}{1+cthx\cdot cthy+cthy\cdot cthz+cthx\cdot cthz}. \qquad (4.19)$$

20) *For every* $x\in\boldsymbol{R}^*$,

$$cth(3x)=\frac{cth^3x+3\cdot cthx}{1+3\cdot cth^2x}. \qquad (4.20)$$

21) *For every* $x\in\boldsymbol{R}$,

$$1-th^2x=\frac{1}{ch^2x}=sch^2x; \qquad (4.21)$$

i.e.:

$sch^2x+th^2x=1;$ (4.21′)

or equivalent:

$$ch^2x=\frac{1}{1-th^2x}=\frac{1}{sch^2x}. \qquad (4.21'')$$

22) *For every* $x\in\boldsymbol{R}^*$,

$$cth^2x-1=\frac{1}{sh^2x}=csh^2x; \qquad (4.22)$$

i.e.:

$cth^2x-csh^2x=1;$ (4.22′)

or equivalent:

$$sh^2x=\frac{1}{cth^2x-1}=\frac{1}{csh^2x}. \qquad (4.22'')$$

23) *For every* $x, y\in\boldsymbol{R}$,

$$shx+shy=2\cdot sh\left(\frac{x+y}{2}\right)\cdot ch\left(\frac{x-y}{2}\right). \qquad (4.23)$$

24) *For every* $x, y\in\boldsymbol{R}$,

$$shx-shy=2\cdot sh\left(\frac{x-y}{2}\right)\cdot ch\left(\frac{x+y}{2}\right). \qquad (4.24)$$

25) *For every* $x, y\in\boldsymbol{R}$,

$$chx+chy=2\cdot ch\left(\frac{x+y}{2}\right)\cdot ch\left(\frac{x-y}{2}\right). \qquad (4.25)$$

26) *For every* $x, y\in\boldsymbol{R}$,

$$chx-chy=2\cdot sh\left(\frac{x+y}{2}\right)\cdot sh\left(\frac{x-y}{2}\right). \qquad (4.26)$$

27) *For every* $x, y\in\boldsymbol{R}$,

$$thx+thy=\frac{sh(x+y)}{chx\cdot chy}. \qquad (4.27)$$

28) *For every* $x, y\in\boldsymbol{R}$,

$$thx-thy=\frac{sh(x-y)}{chx\cdot chy}. \qquad (4.28)$$

29) *For every* $x, y\in\boldsymbol{R}^*$,

$$cthx+cthy=\frac{sh(x+y)}{shx\cdot shy}. \qquad (4.29)$$

30) *For every* $x, y\in\boldsymbol{R}^*$,

$$cthx-cthy=-\frac{sh(x-y)}{shx\cdot shy}. \qquad (4.30)$$

31) *For every* $x, y\in\boldsymbol{R}$,

$$shx\cdot chy=\frac{sh(x+y)+sh(x-y)}{2}. \quad (4.31)$$

32) *For every $x, y\in\mathbf{R}$,*

$$chx\cdot shy=\frac{sh(x+y)-sh(x-y)}{2}. \quad (4.32)$$

33) *For every $x, y\in\mathbf{R}$,*

$$chx\cdot chy=\frac{ch(x+y)+ch(x-y)}{2}. \quad (4.33)$$

34) *For every $x, y\in\mathbf{R}$,*

$$shx\cdot shy=\frac{ch(x+y)-ch(x-y)}{2}. \quad (4.34)$$

35) *For every $x\in\mathbf{R}$,*

$$sh(2x)=\frac{2\cdot thx}{1-th^2x}. \quad (4.35)$$

36) *For every $x\in\mathbf{R}$,*

$$ch(2x)=\frac{1+th^2x}{1-th^2x}. \quad (4.36)$$

37) *For every $x\in\mathbf{R}$,*

$$sh^2x=\frac{1}{2}\cdot ch(2x)-\frac{1}{2}. \quad (4.37)$$

38) *For every $x\in\mathbf{R}$,*

$$ch^2x=\frac{1}{2}\cdot ch(2x)+\frac{1}{2}. \quad (4.38)$$

39) *For every $x\in\mathbf{R}$,*

$$sh^3x=\frac{1}{4}\cdot sh(3x)-\frac{3}{4}\cdot shx. \quad (4.39)$$

40) *For every $x\in\mathbf{R}$,*

$$ch^3x=\frac{1}{4}\cdot ch(3x)+\frac{3}{4}\cdot chx. \quad (4.40)$$

41) *For every $x\in\mathbf{R}$,*

$$sh^4x=\frac{1}{8}\cdot ch(4x)-\frac{1}{2}\cdot ch(2x)+\frac{3}{8}. \quad (4.41)$$

42) *For every $x\in\mathbf{R}$,*

$$ch^4x=\frac{1}{8}\cdot ch(4x)+\frac{1}{2}\cdot ch(2x)+\frac{3}{8}. \quad (4.42)$$

F. Properties immediate of the inverses of hyperbolic functions

43) *For every $x\in\mathbf{R}$,*

$$sh^{-1}(-x)=-sh^{-1}x. \quad (4.43)$$

44) *For every $x\in[1,+\infty)$,*

$$ch_1^{-1}x=-ch_2^{-1}x. \quad (4.44)$$

45) *For every $x\in(-1,1)$,*

$$th^{-1}(-x)=-th^{-1}x. \quad (4.45)$$

46) *For every $x\in(-\infty,-1)\cup(1,+\infty)$,*

$$cth^{-1}(-x)=-cth^{-1}x. \quad (4.46)$$

47) *For every $x\in(0,1]$,*

$$sch_1^{-1}x=-sch_2^{-1}x. \quad (4.47)$$

48) *For every $x\in(-\infty,0)$,*

$$csh_1^{-1}x=-csh_2^{-1}(-x), \quad (4.48)$$

or equivalent, for every $x\in(0,+\infty)$,

$$csh_2^{-1}x=-csh_1^{-1}(-x). \quad (4.48')$$

G. The derivatives of the inverses of hyperbolic functions

49) *For every $x\in\mathbf{R}$,*

$$(sh^{-1}x)'=\frac{1}{\sqrt{x^2+1}}. \quad (4.49)$$

50) *For every $x\in(1,+\infty)$,*

$$(ch_1^{-1}x)'=-\frac{1}{\sqrt{x^2-1}} \quad (4.50)$$

and, for every $x\in(1,+\infty)$,

$$(ch_2^{-1}x)'=\frac{1}{\sqrt{x^2-1}}. \quad (4.50')$$

51) *For every $x\in(-1,1)$,*

$$(th^{-1}x)'=\frac{1}{1-x^2}. \quad (4.51)$$

52) *For every $x\in(-\infty,-1)$,*

$$(cth_1^{-1}x)'=\frac{1}{1-x^2} \quad (4.52)$$

and, for every $x\in(1,+\infty)$,

$$(cth_2^{-1}x)'=\frac{1}{1-x^2}, \quad (4.52')$$

in other words, for every $x\in(-\infty,-1)\cup(1,+\infty)$,

$$(cth^{-1}x)'=\frac{1}{1-x^2}. \quad (4.52'')$$

53) *For every $x\in(0,1)$,*

$$(sch_1^{-1}x)'=\frac{1}{x\cdot\sqrt{1-x^2}} \quad (4.53)$$

and, for every $x\in(0,1)$,

$$(sch_2^{-1}x)'=-\frac{1}{x\cdot\sqrt{1-x^2}}. \quad (4.53')$$

54) *For every $x\in(-\infty,0)$,*

$$(csh_1^{-1}x)'=\frac{1}{x\cdot\sqrt{x^2+1}} \quad (4.54)$$

and, for every $x\in(0,+\infty)$,

$$(csh_2^{-1}x)'=-\frac{1}{x\cdot\sqrt{x^2+1}}. \quad (4.54')$$

***Proof*:** **1)** According to the equality (3.2), for every $x\in\mathbf{R}$,

$$\text{shx}=\text{sh}\left(\frac{x}{2}+\frac{x}{2}\right)=\text{sh}\frac{x}{2}\cdot\text{ch}\frac{x}{2}+\text{sh}\frac{x}{2}\cdot\text{ch}\frac{x}{2}=2\cdot\text{sh}\left(\frac{x}{2}\right)\cdot\text{ch}\left(\frac{x}{2}\right);$$

so, the equality (4.1) holds.

2) According to the equality (3.2), for every $x\in\mathbf{R}$,

$$\text{sh}(2x)=\text{sh}(x+x)=\text{shx}\cdot\text{chx}+\text{shx}\cdot\text{chx}=2\cdot\text{shx}\cdot\text{chx};$$

so, the equality (4.2) holds.

3) According to the equality (3.4), for every $x\in\mathbf{R}$,

$$\text{chx}=\text{ch}\left(\frac{x}{2}+\frac{x}{2}\right)=\text{sh}\left(\frac{x}{2}\right)\cdot\text{sh}\left(\frac{x}{2}\right)+\text{ch}\left(\frac{x}{2}\right)\cdot\text{ch}\left(\frac{x}{2}\right)=\text{sh}^2\left(\frac{x}{2}\right)+\text{ch}^2\left(\frac{x}{2}\right);$$

so, the equality (4.3) holds.

4) According to the equality (3.4), for every $x\in\mathbf{R}$,

$$\text{ch}(2x)=\text{ch}(x+x)=\text{chx}\cdot\text{chx}+\text{shx}\cdot\text{shx}=\text{sh}^2x+\text{ch}^2x;$$

so, the equality (4.4) holds. The equalities (4.4′) and (4.4″) are obtained from the equalities (4.4) and (3.1), replacing one after the other, sh^2x, respective ch^2x.

5) According to the equality (4.3) or to the equality (4.4″), for every $x\in\mathbf{R}$,

$$\text{chx}=2\cdot\text{sh}^2\left(\frac{x}{2}\right)+1,$$

whence, according to Definition 2.1, obtain both equalities (4.5) – for $x<0$, respective (4.5′) – for $x\geq 0$.

6) According to the equality (4.3) or to the equality (4.4′), for every $x\in\mathbf{R}$,

$$\text{chx}=2\cdot\text{ch}^2\left(\frac{x}{2}\right)-1,$$

whence, according to the inequality (2.15), obtain the equality (4.6).

7) According to the equality (3.6), for every $x\in\mathbf{R}$,

$$\text{thx}=\text{th}\left(\frac{x}{2}+\frac{x}{2}\right)=\frac{\text{th}\left(\frac{x}{2}\right)+\text{th}\left(\frac{x}{2}\right)}{1+\text{th}\left(\frac{x}{2}\right)\cdot\text{th}\left(\frac{x}{2}\right)}=\frac{2\cdot\text{th}\left(\frac{x}{2}\right)}{1+\text{th}^2\left(\frac{x}{2}\right)};$$

so, the equality (4.7) holds.

8) According to the equality (3.6), for every $x\in\mathbf{R}$,

$$\text{th}(2x)=\text{th}(x+x)=\frac{\text{thx}+\text{thx}}{1+\text{thx}\cdot\text{thx}}=\frac{2\cdot\text{thx}}{1+\text{th}^2x};$$

so, the equality (4.8) holds.

9) For every $x\in\mathbf{R}$, we have the equalities:

$$\text{th}\left(\frac{x}{2}\right)=\frac{\text{sh}\left(\frac{x}{2}\right)}{\text{ch}\left(\frac{x}{2}\right)} \quad \text{(according to the equality (2.3))}$$

$$=\begin{cases}-\sqrt{\dfrac{\text{chx}-1}{\text{chx}+1}}, \text{if } x<0\\ \sqrt{\dfrac{\text{chx}-1}{\text{chx}+1}}, \text{if } x\geq 0\end{cases} \quad \text{(according to the equalities (4.5), (4.5′) and (4.6))}$$

$$=\frac{2\cdot\text{sh}\left(\frac{x}{2}\right)\cdot\text{ch}\left(\frac{x}{2}\right)}{2\cdot\text{ch}^2\left(\frac{x}{2}\right)}$$

$=\dfrac{\text{shx}}{\text{chx}+1}$ (according to the equalities (4.1) and (4.4′))

$=\dfrac{\text{sh}^2\text{x}}{\text{shx}\cdot(\text{chx}+1)}$

$=\dfrac{\text{ch}^2\text{x}-1}{\text{shx}\cdot(\text{chx}+1)}$ (according to the equality (3.1))

$=\dfrac{(\text{chx}-1)\cdot(\text{chx}+1)}{\text{shx}\cdot(\text{chx}+1)}$ (according to the equality (3.1))

$=\dfrac{\text{chx}-1}{\text{shx}}=\dfrac{\text{chx}}{\text{shx}}-\dfrac{1}{\text{shx}}$

=cthx-cshx (according to the equalities (2.4) and (2.6)).

Therefore, the equalities (4.9) hold.

10) According to the equality (3.8), for every $x\in\mathbf{R}^*$,

$$\text{cthx}=\text{cth}\left(\frac{x}{2}+\frac{x}{2}\right)=\frac{\text{cth}\left(\frac{x}{2}\right)\cdot\text{cth}\left(\frac{x}{2}\right)+1}{\text{cth}\left(\frac{x}{2}\right)+\text{cth}\left(\frac{x}{2}\right)}=\frac{\text{cth}^2\left(\frac{x}{2}\right)+1}{2\cdot\text{cth}\left(\frac{x}{2}\right)};$$

which shows that the equality (4.10) holds.

11) According to the equality (3.8), for every $x\in\mathbf{R}^*$,

$$\text{cth}(2x)=\text{cth}(x+x)=\frac{\text{cthx}\cdot\text{cthx}+1}{\text{cthx}+\text{cthx}}=\frac{\text{cth}^2x+1}{2\cdot\text{cthx}};$$

which shows that the equality (4.11) holds.

12) For every $x\in\mathbf{R}^*$, we have the equalities:

$\text{cth}\left(\frac{x}{2}\right)=\dfrac{\text{ch}\left(\frac{x}{2}\right)}{\text{sh}\left(\frac{x}{2}\right)}$ (according to the equality (2.4))

$=\begin{cases}-\sqrt{\dfrac{\text{chx}+1}{\text{chx}-1}}, & \text{if } x<0\\ \sqrt{\dfrac{\text{chx}+1}{\text{chx}-1}}, & \text{if } x\geq 0\end{cases}$ (according to the equalities (4.5), (4.5′) and (4.6))

$=\dfrac{2\cdot\text{ch}^2\left(\frac{x}{2}\right)}{2\cdot\text{sh}\left(\frac{x}{2}\right)\cdot\text{ch}\left(\frac{x}{2}\right)}$

$=\dfrac{\text{chx}+1}{\text{shx}}$ (according to the equalities (4.1) and (4.4′))

$=\dfrac{(\text{chx}+1)\cdot(\text{chx}-1)}{\text{shx}\cdot(\text{chx}-1)}=\dfrac{\text{ch}^2\text{x}-1}{\text{shx}\cdot(\text{chx}-1)}$

$=\dfrac{\text{sh}^2\text{x}}{\text{shx}\cdot(\text{chx}-1)}$ (according to the equality (3.1))

$=\dfrac{\text{shx}}{\text{chx}-1}=\dfrac{\text{chx}}{\text{shx}}+\dfrac{1}{\text{shx}}$

=cthx+cshx (according to the equalities (2.4) and (2.6)).

Therefore, the equalities (4.12) hold.

13) According to the equalities (3.2) and (3.4), for every x, y, z$\in$**R**,

$$\begin{aligned}\mathrm{sh}(x+y+z)&=\mathrm{sh}[(x+y)+z]=\mathrm{sh}(x+y)\cdot\mathrm{ch}z+\mathrm{sh}z\cdot\mathrm{ch}(x+y)\\&=(\mathrm{sh}x\cdot\mathrm{ch}y+\mathrm{ch}x\cdot\mathrm{sh}y)\cdot\mathrm{ch}z+\mathrm{sh}z\cdot(\mathrm{ch}x\cdot\mathrm{ch}y+\mathrm{sh}x\cdot\mathrm{sh}y)\\&=\mathrm{sh}x\cdot\mathrm{ch}y\cdot\mathrm{ch}z+\mathrm{ch}x\cdot\mathrm{sh}y\cdot\mathrm{ch}z+\mathrm{ch}x\cdot\mathrm{ch}y\cdot\mathrm{sh}z+\mathrm{sh}x\cdot\mathrm{sh}y\cdot\mathrm{sh}z;\end{aligned}$$

whence it follows that the equality (4.13) holds.

14) According to the equalities (4.13) and (3.1), for every x$\in$**R**,

$$\begin{aligned}\mathrm{sh}(3x)&=\mathrm{sh}(x+x+x)=\mathrm{sh}x\cdot\mathrm{ch}x\cdot\mathrm{ch}x+\mathrm{ch}x\cdot\mathrm{sh}x\cdot\mathrm{ch}x+\mathrm{ch}x\cdot\mathrm{ch}x\cdot\mathrm{sh}x+\mathrm{sh}x\cdot\mathrm{sh}x\cdot\mathrm{sh}x\\&=3\cdot\mathrm{sh}x\cdot\mathrm{ch}^2x+\mathrm{sh}^3x=\mathrm{sh}x\cdot(3\cdot\mathrm{ch}^2x+\mathrm{sh}^2x)=\mathrm{sh}x\cdot[3\cdot(1+\mathrm{sh}^2x)+\mathrm{sh}^2x]\\&=\mathrm{sh}x\cdot(4\cdot\mathrm{sh}^2x+3)=\mathrm{sh}x\cdot[4\cdot(\mathrm{ch}^2x-1)+3]=\mathrm{sh}x\cdot(4\cdot\mathrm{ch}^2x-1);\end{aligned}$$

i.e., we have shown that the equalities (4.14) hold.

15) According to the equalities (3.4) and (3.2), for every x, y, z$\in$**R**,

$$\begin{aligned}\mathrm{ch}(x+y+z)&=\mathrm{ch}[(x+y)+z]=\mathrm{ch}(x+y)\cdot\mathrm{ch}z+\mathrm{sh}z\cdot\mathrm{sh}(x+y)\\&=(\mathrm{ch}x\cdot\mathrm{ch}y+\mathrm{sh}x\cdot\mathrm{sh}y)\cdot\mathrm{ch}z+\mathrm{sh}z\cdot(\mathrm{sh}x\cdot\mathrm{ch}y+\mathrm{ch}x\cdot\mathrm{sh}y)\\&=\mathrm{ch}x\cdot\mathrm{ch}y\cdot\mathrm{ch}z+\mathrm{sh}x\cdot\mathrm{sh}y\cdot\mathrm{ch}z+\mathrm{sh}x\cdot\mathrm{ch}y\cdot\mathrm{sh}z+\mathrm{ch}x\cdot\mathrm{sh}y\cdot\mathrm{sh}z;\end{aligned}$$

which shows that the equality (4.15) holds.

16) According to the equalities (4.15) and (3.1), for every x$\in$**R**,

$$\begin{aligned}\mathrm{ch}(3x)&=\mathrm{ch}(x+x+x)=\mathrm{ch}x\cdot\mathrm{ch}x\cdot\mathrm{ch}x+\mathrm{sh}x\cdot\mathrm{sh}x\cdot\mathrm{ch}x+\mathrm{sh}x\cdot\mathrm{ch}x\cdot\mathrm{sh}x+\mathrm{ch}x\cdot\mathrm{sh}x\cdot\mathrm{sh}x\\&=\mathrm{ch}^3x+3\mathrm{sh}^2x\cdot\mathrm{ch}x=\mathrm{ch}x\cdot(\mathrm{ch}^2x+3\mathrm{sh}^2x)=\mathrm{ch}x\cdot[3\cdot(\mathrm{ch}^2x-1)+\mathrm{ch}^2x]\\&=\mathrm{ch}x\cdot(4\cdot\mathrm{ch}^2x-3)=\mathrm{ch}x\cdot[4\cdot(\mathrm{sh}^2x+1)-3]=\mathrm{ch}x\cdot(4\cdot\mathrm{sh}^2x+1);\end{aligned}$$

i.e., we have shown that the equalities (4.16) hold.

17) According to the equalities (3.6) and (2.3), for every x, y, z$\in$**R**,

$$\mathrm{th}(x+y+z)=\mathrm{th}[(x+y)+z]=\frac{\mathrm{th}(x+y)+\mathrm{th}z}{1+\mathrm{th}(x+y)\cdot\mathrm{th}z}=\frac{\frac{\mathrm{th}x+\mathrm{th}y}{1+\mathrm{th}x\cdot\mathrm{th}y}+\mathrm{th}z}{1+\frac{\mathrm{th}x+\mathrm{th}y}{1+\mathrm{th}x\cdot\mathrm{th}y}\cdot\mathrm{th}z}$$

$$=\frac{\frac{\mathrm{th}x+\mathrm{th}y+\mathrm{th}z+\mathrm{th}x\cdot\mathrm{th}y\cdot\mathrm{th}z}{1+\mathrm{th}x\cdot\mathrm{th}y}}{\frac{1+\mathrm{th}x\cdot\mathrm{th}y+\mathrm{th}y\cdot\mathrm{th}z+\mathrm{th}x\cdot\mathrm{th}z}{1+\mathrm{th}x\cdot\mathrm{th}y}}=\frac{\mathrm{th}x+\mathrm{th}y+\mathrm{th}z+\mathrm{th}x\cdot\mathrm{th}y\cdot\mathrm{th}z}{1+\mathrm{th}x\cdot\mathrm{th}y+\mathrm{th}y\cdot\mathrm{th}z+\mathrm{th}x\cdot\mathrm{th}z}.$$

***Otherwise*:** According to the equalities (2.3), (4.13) and (4.15), for every x, y, z$\in$**R**,

$$\mathrm{th}(x+y+z)=\frac{\mathrm{sh}(x+y+z)}{\mathrm{ch}(x+y+z)}$$

$$=\frac{\mathrm{sh}x\cdot\mathrm{ch}y\cdot\mathrm{ch}z+\mathrm{ch}x\cdot\mathrm{sh}y\cdot\mathrm{ch}z+\mathrm{ch}x\cdot\mathrm{ch}y\cdot\mathrm{sh}z+\mathrm{ch}x\cdot\mathrm{ch}y\cdot\mathrm{ch}z}{\mathrm{ch}x\cdot\mathrm{ch}y\cdot\mathrm{ch}z+\mathrm{sh}x\cdot\mathrm{sh}y\cdot\mathrm{ch}z+\mathrm{sh}x\cdot\mathrm{ch}y\cdot\mathrm{sh}z+\mathrm{ch}x\cdot\mathrm{sh}y\cdot\mathrm{sh}z}$$

$$=\frac{\frac{\mathrm{sh}x\cdot\mathrm{ch}y\cdot\mathrm{ch}z+\mathrm{ch}x\cdot\mathrm{sh}y\cdot\mathrm{ch}z+\mathrm{ch}x\cdot\mathrm{ch}y\cdot\mathrm{sh}z+\mathrm{ch}x\cdot\mathrm{ch}y\cdot\mathrm{ch}z}{\mathrm{ch}x\cdot\mathrm{ch}y\cdot\mathrm{ch}z}}{\frac{\mathrm{ch}x\cdot\mathrm{ch}y\cdot\mathrm{ch}z+\mathrm{sh}x\cdot\mathrm{sh}y\cdot\mathrm{ch}z+\mathrm{sh}x\cdot\mathrm{ch}y\cdot\mathrm{sh}z+\mathrm{ch}x\cdot\mathrm{sh}y\cdot\mathrm{sh}z}{\mathrm{ch}x\cdot\mathrm{ch}y\cdot\mathrm{ch}z}}$$

$$=\frac{\frac{\mathrm{sh}x\cdot\mathrm{ch}y\cdot\mathrm{ch}z}{\mathrm{ch}x\cdot\mathrm{ch}y\cdot\mathrm{ch}z}+\frac{\mathrm{ch}x\cdot\mathrm{sh}y\cdot\mathrm{ch}z}{\mathrm{ch}x\cdot\mathrm{ch}y\cdot\mathrm{ch}z}+\frac{\mathrm{ch}x\cdot\mathrm{ch}y\cdot\mathrm{sh}z}{\mathrm{ch}x\cdot\mathrm{ch}y\cdot\mathrm{ch}z}+\frac{\mathrm{ch}x\cdot\mathrm{ch}y\cdot\mathrm{ch}z}{\mathrm{ch}x\cdot\mathrm{ch}y\cdot\mathrm{ch}z}}{\frac{\mathrm{ch}x\cdot\mathrm{ch}y\cdot\mathrm{ch}z}{\mathrm{ch}x\cdot\mathrm{ch}y\cdot\mathrm{ch}z}+\frac{\mathrm{sh}x\cdot\mathrm{sh}y\cdot\mathrm{ch}z}{\mathrm{ch}x\cdot\mathrm{ch}y\cdot\mathrm{ch}z}+\frac{\mathrm{sh}x\cdot\mathrm{ch}y\cdot\mathrm{sh}z}{\mathrm{ch}x\cdot\mathrm{ch}y\cdot\mathrm{ch}z}+\frac{\mathrm{ch}x\cdot\mathrm{sh}y\cdot\mathrm{ch}z}{\mathrm{ch}x\cdot\mathrm{ch}y\cdot\mathrm{ch}z}}$$

$$=\frac{\mathrm{th}x+\mathrm{th}y+\mathrm{th}z+\mathrm{th}x\cdot\mathrm{th}y\cdot\mathrm{th}z}{1+\mathrm{th}x\cdot\mathrm{th}y+\mathrm{th}y\cdot\mathrm{th}z+\mathrm{th}x\cdot\mathrm{th}z};$$

which shows that the equality (4.17) holds. Because, for every x, y, z$\in$**R**, th(x+y) and thz$\in$(-1,1), it follows that:

$\mathrm{th}(x+y)\cdot\mathrm{th}z+1\neq0$,

i.e.:

$1+\mathrm{th}x\cdot\mathrm{th}y+\mathrm{th}y\cdot\mathrm{th}z+\mathrm{th}x\cdot\mathrm{th}z\neq0$.

18) According to the equality (4.17), for every $x\in\mathbf{R}$,

$$\mathrm{th}(3x)=\mathrm{th}(x+x+x)=\frac{\mathrm{th}x+\mathrm{th}x+\mathrm{th}x+\mathrm{th}x\cdot\mathrm{th}x\cdot\mathrm{th}x}{1+\mathrm{th}x\cdot\mathrm{th}x+\mathrm{th}x\cdot\mathrm{th}x+\mathrm{th}x\cdot\mathrm{th}x}=\frac{3\cdot\mathrm{th}x+\mathrm{th}^3x}{1+3\cdot\mathrm{th}^2x}.$$

Otherwise: According to the equalities (2.3), (4.14) and (4.16), for every $x\in\mathbf{R}$,

$$\mathrm{th}(3x)=\frac{\mathrm{sh}(3x)}{\mathrm{ch}(3x)}=\frac{\mathrm{sh}x\cdot(3\cdot\mathrm{ch}^2x+\mathrm{sh}^2x)}{\mathrm{ch}x\cdot(\mathrm{ch}^2x+3\mathrm{sh}^2x)}=\frac{\dfrac{\mathrm{sh}x\cdot(3\cdot\mathrm{ch}^2y+\mathrm{sh}^2x)}{\mathrm{ch}^3x}}{\dfrac{\mathrm{ch}x\cdot(\mathrm{ch}^2y+3\cdot\mathrm{sh}^2x)}{\mathrm{ch}^3x}}=\frac{\dfrac{\mathrm{sh}x}{\mathrm{ch}x}\cdot\left(3\cdot\dfrac{\mathrm{ch}^2x}{\mathrm{ch}^2x}+\dfrac{\mathrm{sh}^2x}{\mathrm{ch}^2x}\right)}{\dfrac{\mathrm{ch}x}{\mathrm{ch}x}\cdot\left(\dfrac{\mathrm{ch}^2x}{\mathrm{ch}^2x}+3\cdot\dfrac{\mathrm{sh}^2x}{\mathrm{ch}^2x}\right)}$$

$$=\frac{\mathrm{th}x\cdot(3+\mathrm{th}^2x)}{1+3\cdot\mathrm{th}^2x};$$

which shows that the equality (4.18) holds.

19) According to the equalities (3.8) and (2.4), for every $x, y, z\in\mathbf{R}^*$, such that $x+y$, $y+z$, $x+z$, $x+y+z\in\mathbf{R}^*$,

$$\mathrm{cth}(x+y+z)=\mathrm{cth}[(x+y)+z]=\frac{\mathrm{cth}(x+y)\cdot\mathrm{cth}z+1}{\mathrm{cth}(x+y)+\mathrm{cth}z}=\frac{\dfrac{\mathrm{cth}x\cdot\mathrm{cth}y+1}{\mathrm{cth}x+\mathrm{cth}y}\cdot\mathrm{cth}z+1}{\dfrac{\mathrm{cth}x\cdot\mathrm{cth}y+1}{\mathrm{cth}x+\mathrm{cth}y}+\mathrm{cth}z}$$

$$=\frac{\dfrac{\mathrm{cth}x\cdot\mathrm{cth}y\cdot\mathrm{cth}z+\mathrm{cth}x+\mathrm{cth}y+\mathrm{cth}z}{\mathrm{cth}x+\mathrm{cth}y}}{\dfrac{\mathrm{cth}x\cdot\mathrm{cth}y+\mathrm{cth}y\cdot\mathrm{cth}z+\mathrm{cth}x\cdot\mathrm{cth}z+1}{\mathrm{cth}x+\mathrm{cth}y}}=\frac{\mathrm{cth}x\cdot\mathrm{cth}y\cdot\mathrm{cth}z+\mathrm{cth}x+\mathrm{cth}y+\mathrm{cth}z}{1+\mathrm{cth}x\cdot\mathrm{cth}y+\mathrm{cth}y\cdot\mathrm{cth}z+\mathrm{cth}x\cdot\mathrm{cth}z}.$$

Otherwise: According to the equalities (2.4), (4.13) and (4.15), for every $x, y, z\in\mathbf{R}^*$, such that $x+y$, $y+z$, $x+z$, $x+y+z\in\mathbf{R}^*$,

$$\mathrm{cth}(x+y+z)=\frac{\mathrm{ch}(x+y+z)}{\mathrm{sh}(x+y+z)}$$

$$=\frac{\mathrm{ch}x\cdot\mathrm{ch}y\cdot\mathrm{ch}z+\mathrm{sh}x\cdot\mathrm{sh}y\cdot\mathrm{ch}z+\mathrm{sh}x\cdot\mathrm{ch}y\cdot\mathrm{sh}z+\mathrm{ch}x\cdot\mathrm{sh}y\cdot\mathrm{sh}z}{\mathrm{sh}x\cdot\mathrm{ch}y\cdot\mathrm{ch}z+\mathrm{ch}x\cdot\mathrm{sh}y\cdot\mathrm{ch}z+\mathrm{ch}x\cdot\mathrm{ch}y\cdot\mathrm{sh}z+\mathrm{ch}x\cdot\mathrm{ch}y\cdot\mathrm{ch}z}$$

$$=\frac{\dfrac{\mathrm{ch}x\cdot\mathrm{ch}y\cdot\mathrm{ch}z+\mathrm{sh}x\cdot\mathrm{sh}y\cdot\mathrm{ch}z+\mathrm{sh}x\cdot\mathrm{ch}y\cdot\mathrm{sh}z+\mathrm{ch}x\cdot\mathrm{sh}y\cdot\mathrm{sh}z}{\mathrm{sh}x\cdot\mathrm{sh}y\cdot\mathrm{sh}z}}{\dfrac{\mathrm{sh}x\cdot\mathrm{ch}y\cdot\mathrm{ch}z+\mathrm{ch}x\cdot\mathrm{sh}y\cdot\mathrm{ch}z+\mathrm{ch}x\cdot\mathrm{ch}y\cdot\mathrm{sh}z+\mathrm{ch}x\cdot\mathrm{ch}y\cdot\mathrm{ch}z}{\mathrm{sh}x\cdot\mathrm{sh}y\cdot\mathrm{sh}z}}$$

$$=\frac{\dfrac{\mathrm{ch}x\cdot\mathrm{ch}y\cdot\mathrm{ch}z}{\mathrm{sh}x\cdot\mathrm{sh}y\cdot\mathrm{sh}z}+\dfrac{\mathrm{sh}x\cdot\mathrm{sh}y\cdot\mathrm{ch}z}{\mathrm{sh}x\cdot\mathrm{sh}y\cdot\mathrm{sh}z}+\dfrac{\mathrm{sh}x\cdot\mathrm{ch}y\cdot\mathrm{sh}z}{\mathrm{sh}x\cdot\mathrm{sh}y\cdot\mathrm{sh}z}+\dfrac{\mathrm{ch}x\cdot\mathrm{sh}y\cdot\mathrm{ch}z}{\mathrm{sh}x\cdot\mathrm{sh}y\cdot\mathrm{sh}z}}{\dfrac{\mathrm{sh}x\cdot\mathrm{ch}y\cdot\mathrm{ch}z}{\mathrm{sh}x\cdot\mathrm{sh}y\cdot\mathrm{sh}z}+\dfrac{\mathrm{ch}x\cdot\mathrm{sh}y\cdot\mathrm{ch}z}{\mathrm{sh}x\cdot\mathrm{sh}y\cdot\mathrm{sh}z}+\dfrac{\mathrm{sh}x\cdot\mathrm{ch}y\cdot\mathrm{sh}z}{\mathrm{sh}x\cdot\mathrm{sh}y\cdot\mathrm{sh}z}+\dfrac{\mathrm{ch}x\cdot\mathrm{sh}y\cdot\mathrm{sh}z}{\mathrm{sh}x\cdot\mathrm{sh}y\cdot\mathrm{sh}z}}$$

$$=\frac{\mathrm{cth}x\cdot\mathrm{cth}y\cdot\mathrm{cth}z+\mathrm{cth}x+\mathrm{cth}y+\mathrm{cth}z}{1+\mathrm{cth}x\cdot\mathrm{cth}y+\mathrm{cth}y\cdot\mathrm{cth}z+\mathrm{cth}x\cdot\mathrm{cth}z}.$$

Otherwise: According to the equalities (2.4) and (4.17), for every $x, y, z\in\mathbf{R}^*$, such that $x+y$, $y+z$, $x+z$, $x+y+z\in\mathbf{R}^*$,

$$\mathrm{cth}(x+y+z)=\frac{1}{\mathrm{th}(x+y+z)}=\frac{1+\mathrm{th}x\cdot\mathrm{th}y+\mathrm{th}y\cdot\mathrm{th}z+\mathrm{th}x\cdot\mathrm{th}z}{\mathrm{th}x+\mathrm{th}y+\mathrm{th}z+\mathrm{th}x\cdot\mathrm{th}y\cdot\mathrm{th}z}$$

$$=\frac{1+\dfrac{1}{\mathrm{cth}x}\cdot\dfrac{1}{\mathrm{cth}y}+\dfrac{1}{\mathrm{cth}y}\cdot\dfrac{1}{\mathrm{cth}z}+\dfrac{1}{\mathrm{cth}x}\cdot\dfrac{1}{\mathrm{cth}z}}{\dfrac{1}{\mathrm{cth}x}+\dfrac{1}{\mathrm{cth}y}+\dfrac{1}{\mathrm{cth}z}+\dfrac{1}{\mathrm{cth}x}\cdot\dfrac{1}{\mathrm{cth}y}\cdot\dfrac{1}{\mathrm{cth}z}}$$

$$=\frac{\dfrac{\mathrm{cthx}\cdot\mathrm{cthy}\cdot\mathrm{cthz}+\mathrm{cthx}+\mathrm{cthy}+\mathrm{cthz}}{\mathrm{cthx}\cdot\mathrm{cthy}\cdot\mathrm{cthz}}}{\dfrac{1+\mathrm{cthx}\cdot\mathrm{cthy}+\mathrm{cthy}\cdot\mathrm{cthz}+\mathrm{cthx}\cdot\mathrm{cthz}}{\mathrm{cthx}\cdot\mathrm{cthy}\cdot\mathrm{cthz}}}$$

$$=\frac{\mathrm{cthx}\cdot\mathrm{cthy}\cdot\mathrm{cthz}+\mathrm{cthx}+\mathrm{cthy}+\mathrm{cthz}}{1+\mathrm{cthx}\cdot\mathrm{cthy}+\mathrm{cthy}\cdot\mathrm{cthz}+\mathrm{cthx}\cdot\mathrm{cthz}};$$

therefore, the equality (4.19) holds. Then, since, according to the hypothesis,

$\mathrm{cth}(x+y)+\mathrm{cthz}\neq 0$,

it follows that:

$1+\mathrm{cthx}\cdot\mathrm{cthy}+\mathrm{cthy}\cdot\mathrm{cthz}+\mathrm{cthx}\cdot\mathrm{cthz}\neq 0$.

20) According to the equality (4.17), for every $x\in\mathbf{R}^*$,

$$\mathrm{cth}(3x)=\mathrm{cth}(x+x+x)=\frac{\mathrm{cthx}\cdot\mathrm{cthx}\cdot\mathrm{cthx}+\mathrm{cthx}+\mathrm{cthx}+\mathrm{cthx}}{1+\mathrm{cthx}\cdot\mathrm{cthx}+\mathrm{cthx}\cdot\mathrm{cthx}+\mathrm{cthx}\cdot\mathrm{cthx}}=\frac{\mathrm{cth}^3x+3\cdot\mathrm{cthx}}{1+3\cdot\mathrm{cth}^2x}.$$

***Otherwise*:** According to the equalities (2.4), (4.14) and (4.16), for every $x\in\mathbf{R}^*$,

$$\mathrm{cth}(3x)=\frac{\mathrm{ch}(3x)}{\mathrm{sh}(3x)}=\frac{\mathrm{chx}\cdot(\mathrm{ch}^2x+3\cdot\mathrm{sh}^2x)}{\mathrm{shx}\cdot(\mathrm{sh}^2x+3\cdot\mathrm{ch}^2x)}=\frac{\dfrac{\mathrm{chx}\cdot(\mathrm{ch}^2x+3\cdot\mathrm{sh}^2x)}{\mathrm{sh}^3x}}{\dfrac{\mathrm{shx}\cdot(\mathrm{sh}^2x+3\cdot\mathrm{ch}^2x)}{\mathrm{sh}^3x}}=\frac{\dfrac{\mathrm{chx}}{\mathrm{shx}}\cdot\left(\dfrac{\mathrm{ch}^2x}{\mathrm{sh}^2x}+3\cdot\dfrac{\mathrm{sh}^2x}{\mathrm{sh}^2x}\right)}{\dfrac{\mathrm{shx}}{\mathrm{shx}}\cdot\left(\dfrac{\mathrm{sh}^2x}{\mathrm{sh}^2x}+3\cdot\dfrac{\mathrm{ch}^2x}{\mathrm{sh}^2x}\right)}$$

$$=\frac{\mathrm{cthx}\cdot(\mathrm{cth}^2x+3)}{1+3\cdot\mathrm{cth}^2x}.$$

***Otherwise*:** According to the equalities (2.4) and (4.18), for every $x\in\mathbf{R}^*$,

$$\mathrm{cth}(3x)=\frac{1}{\mathrm{th}(3x)}=\frac{1+3\cdot\mathrm{th}^2x}{\mathrm{thx}\cdot(3+\mathrm{th}^2x)}=\frac{1+3\cdot\dfrac{1}{\mathrm{cth}^2x}}{\dfrac{1}{\mathrm{cthx}}\cdot\left(3+\dfrac{1}{\mathrm{cth}^2x}\right)}=\frac{\dfrac{\mathrm{cth}^2x+3}{\mathrm{cth}^2x}}{\dfrac{3\cdot\mathrm{cth}^2x+1}{\mathrm{cth}^3x}}=\frac{\mathrm{cthx}\cdot(\mathrm{cth}^2x+3)}{1+3\cdot\mathrm{cth}^2x};$$

therefore, the equality (4.20) holds.

21) According to the equalities (2.3), (3.1) and (2.5), for every $x\in\mathbf{R}$,

$$1-\mathrm{th}^2x=1-\frac{\mathrm{sh}^2x}{\mathrm{ch}^2x}=\frac{\mathrm{ch}^2x-\mathrm{sh}^2x}{\mathrm{ch}^2x}=\frac{1}{\mathrm{ch}^2x}=\mathrm{sch}^2x;$$

whence, according to the same equalities, it follows that, for every $x\in\mathbf{R}$:

$$\mathrm{ch}^2x=\frac{1}{1-\mathrm{th}^2x}=\frac{1}{\mathrm{sch}^2x};$$

therefore, the equalities (4.21), (4.21′) and (4.21″) hold.

22) According to the equalities (2.4), (3.1) and (2.6), for every $x\in\mathbf{R}^*$,

$$\mathrm{cth}^2x-1=\frac{\mathrm{ch}^2x}{\mathrm{sh}^2x}-1=\frac{\mathrm{ch}^2x-\mathrm{sh}^2x}{\mathrm{sh}^2x}=\frac{1}{\mathrm{sh}^2x}=\mathrm{csh}^2x;$$

whence, according to the same equalities, it follows that, for every $x\in\mathbf{R}^*$:

$$\mathrm{sh}^2x=\frac{1}{\mathrm{cth}^2x-1}=\frac{1}{\mathrm{csh}^2x};$$

therefore, the equalities (4.22), (4.22′) and (4.22″) hold.

23) According to the equalities (3.2) and (3.3), for every $a, b\in\mathbf{R}$,

$\mathrm{sh}(a+b)=\mathrm{sha}\cdot\mathrm{chb}+\mathrm{shb}\cdot\mathrm{cha}$ and $\mathrm{sh}(a-b)=\mathrm{sha}\cdot\mathrm{chb}-\mathrm{shb}\cdot\mathrm{cha}$.

By adding together these two relationships, obtain:

(1) $\mathrm{sh}(a+b)+\mathrm{sh}(a-b)=2\cdot\mathrm{sha}\cdot\mathrm{chb}$.

Denoting:

$a+b=x$ and $a-b=y$,

then:

$a=\frac{x+y}{2}$ and $b=\frac{x-y}{2}$,

and the equality (1) becomes the equality (4.23).

24) According to the equalities (3.2) and (3.3), for every a, b$\in$**R**,

sh(a+b)=sha·chb+shb·cha and sh(a-b)=sha·chb-shb·cha.

By decreasing these two relationships, obtain:

(1) sh(a+b)-sh(a-b)=2·shb·cha.

Denoting, again:

a+b=x and a-b=y,

then:

$a=\frac{x+y}{2}$ and $b=\frac{x-y}{2}$,

and the equality (1) becomes the equality (4.24).

25) According to the equalities (3.4) and (3.5), for every a, b$\in$**R**,

ch(a+b)=cha·chb+sha·shb and ch(a-b)=cha·chb-sha·shb.

By adding together these two relationships, obtain:

(1) ch(a+b)+ch(a-b)=2·cha·chb.

Denoting, again:

a+b=x and a-b=y,

then:

$a=\frac{x+y}{2}$ and $b=\frac{x-y}{2}$,

and the equality (1) becomes the equality (4.25).

26) According to the equalities (3.4) and (3.5), for every a, b$\in$**R**,

ch(a+b)=cha·chb+sha·shb and ch(a-b)=cha·chb-sha·shb.

By decreasing these two relationships, obtain:

(1) ch(a+b)-ch(a-b)=2·sha·shb.

Denoting, again:

a+b=x and a-b=y,

then:

$a=\frac{x+y}{2}$ and $b=\frac{x-y}{2}$,

and the equality (1) becomes the equality (4.26).

27) According to the equalities (2.3) and (3.2), for every x, y$\in$**R**,

$$\text{thx+thy}=\frac{\text{shx}}{\text{chx}}+\frac{\text{shy}}{\text{chy}}=\frac{\text{shx}\cdot\text{chy}+\text{shy}\cdot\text{chx}}{\text{chx}\cdot\text{chy}}=\frac{\text{sh}(x+y)}{\text{chx}\cdot\text{chy}};$$

so, the equality (4.27) holds.

28) According to the equalities (2.3) and (3.3), for every x$\in$**R**,

$$\text{thx-thy}=\frac{\text{shx}}{\text{chx}}-\frac{\text{shy}}{\text{chy}}=\frac{\text{shx}\cdot\text{chy}-\text{shy}\cdot\text{chx}}{\text{chx}\cdot\text{chy}}=\frac{\text{sh}(x-y)}{\text{chx}\cdot\text{chy}};$$

so, the equality (4.28) holds.

29) According to the equalities (2.4) and (3.2), for every x, y$\in$**R***,

$$\text{cthx+cthy}=\frac{\text{chx}}{\text{shx}}+\frac{\text{chy}}{\text{shy}}=\frac{\text{shy}\cdot\text{chx}+\text{shx}\cdot\text{chy}}{\text{shx}\cdot\text{shy}}=\frac{\text{sh}(x+y)}{\text{shx}\cdot\text{shy}};$$

so, the equality (4.29) holds.

30) According to the equalities (2.4) and (3.3), for every x, y$\in$**R***,

$$\text{cthx-cthy}=\frac{\text{chx}}{\text{shx}}-\frac{\text{chy}}{\text{shy}}=\frac{\text{shy}\cdot\text{chx}-\text{shx}\cdot\text{chy}}{\text{shx}\cdot\text{shy}}=\frac{\text{sh}(y-x)}{\text{shx}\cdot\text{shy}}=-\frac{\text{sh}(x-y)}{\text{shx}\cdot\text{shy}};$$

so, the equality (4.30) holds.

31) The assertion from the statement follows from the proof of point 23) - the equality (1).
32) The assertion from the statement follows from the proof of point 24) - the equality (1).
33) The assertion from the statement follows from the proof of point 25) - the equality (1).
34) The assertion from the statement follows from the proof of point 26) - the equality (1).
35) For every $x\in\mathbf{R}$, we have the equalities:

$sh(2x)=2\cdot shx\cdot chx$ (according to the equality (4.2))

$=2\cdot\frac{shx}{chx}\cdot ch^2x$

$=2\cdot\frac{shx}{chx}\cdot\frac{1}{1-th^2x}$ (according to the first equality from (4.21))

$=\frac{2\cdot thx}{1-th^2x}$ (according to the equality (2.3));

which shows that the equality (4.35) holds also.
36) For every $x\in\mathbf{R}$, we have the equalities:

$ch(2x)=\frac{ch^2x+sh^2x}{1}$ (according to the equality (4.4))

$=\frac{ch^2x+sh^2x}{ch^2x-sh^2x}$ (according to the equality (3.1))

$$=\frac{\frac{ch^2x+sh^2x}{ch^2x}}{\frac{ch^2x-sh^2x}{ch^2x}}=\frac{1+\frac{sh^2x}{ch^2x}}{1-\frac{sh^2x}{ch^2x}}$$

$=\frac{1+th^2x}{1-th^2x}$ (according to the equality (2.3));

which shows that the equality (4.36) holds also.
37) For every $x\in\mathbf{R}$, the equality (4.37) follows from the equality (4.4″).
38) For every $x\in\mathbf{R}$, the equality (4.38) follows from the equality (4.4′).
39) For every $x\in\mathbf{R}$, the equality (4.39) follows from first equality from (4.14).
40) For every $x\in\mathbf{R}$, the equality (4.40) follows from first equality from (4.16).
41) For every $x\in\mathbf{R}$, we have the equalities:

$ch(4x)=2\cdot sh^2(2x)+1$ (according to the equality (4.4″))

$=2\cdot(2\cdot shx\cdot chx)^2+1$ (according to the equality (4.2))

$=2\cdot 4\cdot sh^2x\cdot ch^2x+1$

$=8\cdot sh^2x\cdot(sh^2x+1)+1$ (according to the equality (3.1))

$=8\cdot sh^4x+8\cdot sh^2x+1$

$=8\cdot sh^4x+8\cdot\left(\frac{1}{2}\cdot ch(2x)-\frac{1}{2}\right)+1$ (according to the equality (4.37))

$=8\cdot sh^4x+8\cdot ch(2x)-3$;

whence it follows that, for every $x\in\mathbf{R}$,

$sh^4x=\frac{1}{8}\cdot ch(4x)-\frac{1}{2}\cdot ch(2x)+\frac{3}{8}$;

so the equality (4.41) holds.
42) For every $x\in\mathbf{R}$, we have the equalities:

$ch(4x)=2\cdot ch^2(2x)-1$ (according to the equality (4.4′))

$=2\cdot(2\cdot ch^2x-1)^2-1$ (according to the equality (4.4′))

$=2\cdot(4\cdot ch^4x-4\cdot ch^2x+1)-1=8\cdot ch^4x-8\cdot ch^2x+1$

$=8\cdot ch^4x-8\cdot\left(\frac{1}{2}\cdot ch(2x)+\frac{1}{2}\right)+1$ (according to the equality (4.38))

$=8\cdot\text{ch}^4x-4\cdot\text{ch}(2x)-3;$

whence it follows that, for every $x\in\mathbf{R}$,

$\text{ch}^4x=\frac{1}{8}\cdot\text{ch}(4x)+\frac{1}{2}\cdot\text{ch}(2x)+\frac{3}{8}$;

so, the equality (4.42) holds.

43) According to the equality (3.22), for every $x\in\mathbf{R}$,

$$\text{sh}^{-1}(-x)=\ln(-x+\sqrt{(-x)^2+1})=\ln(-x+\sqrt{x^2+1})=\ln\left(\frac{1}{\sqrt{x^2+1}+x}\right)=-\ln(x+\sqrt{x^2+1})$$

$$=-\text{sh}^{-1}x;$$

which shows that the equality (4.43) holds.

44) According to the equalities (3.23) and (3.24), for every $x\in[1,+\infty)$,

$$\text{ch}_1^{-1}(-x)=\ln(x-\sqrt{x^2-1})=\ln\left(\frac{x^2-x^2+1}{x+\sqrt{x^2-1}}\right)=\ln\left(\frac{1}{x+\sqrt{x^2-1}}\right)=-\ln(x+\sqrt{x^2-1})$$

$$=-\text{ch}_2^{-1}x;$$

which shows that the equality (4.44) holds.

45) According to the equality (3.25), for every $x\in(-1,1)$,

$$\text{th}^{-1}(-x)=\frac{1}{2}\cdot\ln\frac{1+(-x)}{1-(-x)}=\frac{1}{2}\cdot\ln\frac{1-x}{1+x}=-\frac{1}{2}\cdot\ln\frac{1+x}{1-x}=-\text{th}^{-1}x;$$

which shows that the equality (4.45) holds.

46) According to the equality (3.26″), for every $x\in(-\infty,-1)\cup(1,+\infty)$,

$$\text{cth}^{-1}(-x)=\frac{1}{2}\cdot\ln\frac{(-x)+1}{(-x)-1}=\frac{1}{2}\cdot\ln\frac{-x+1}{-x-1}=\frac{1}{2}\cdot\ln\frac{x-1}{x+1}=-\frac{1}{2}\cdot\ln\frac{x+1}{x-1}=-\text{cth}^{-1}x;$$

which shows that the equality (4.46) holds.

47) According to the equalities (3.27) and (3.28), for every $x\in(0,1]$,

$$\text{sch}_1^{-1}(-x)=\ln\left(\frac{1-\sqrt{1-(-x)^2}}{-x}\right)=\ln\left(\frac{1-\sqrt{1-x^2}}{x}\right)=\ln\left(\frac{x}{1+\sqrt{1-x^2}}\right)=-\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)$$

$$=-\text{sch}_2^{-1}x;$$

which shows that the equality (4.47) holds.

48) According to the equalities (3.29) and (3.30), for every $x\in(-\infty,0)$,

$$\text{csh}_1^{-1}x=\ln\left(\frac{1-\sqrt{1+x^2}}{x}\right)=\ln\left(\frac{-x}{1+\sqrt{1+x^2}}\right)=\ln\left(\frac{-x}{1+\sqrt{(-x)^2+1}}\right)=-\ln\left(\frac{1+\sqrt{(-x)^2+1}}{-x}\right)$$

$$=-\text{csh}_2^{-1}(-x);$$

which shows that the equality (4.48) holds. On the other hand, (again) according to the equalities (3.29) and (3.30), for every $x\in(0,+\infty)$,

$$\text{csh}_2^{-1}x=\ln\left(\frac{1+\sqrt{1+x^2}}{x}\right)=\ln\left(\frac{-x}{1-\sqrt{1+x^2}}\right)=\ln\left(\frac{-x}{1-\sqrt{(-x)^2+1}}\right)=-\ln\left(\frac{1-\sqrt{(-x)^2+1}}{-x}\right)$$

$$=-\text{csh}_1^{-1}(-x);$$

which shows that the equality (4.48′) holds.

49) According to the equality (3.22), for every $x\in\mathbf{R}$,

$$(\text{sh}^{-1}x)'=(\ln(x+\sqrt{x^2+1}))'=\frac{1+\frac{x}{\sqrt{x^2+1}}}{x+\sqrt{x^2+1}}=\frac{\frac{x+\sqrt{x^2+1}}{\sqrt{x^2+1}}}{x+\sqrt{x^2+1}}=\frac{1}{\sqrt{x^2+1}};$$

so, the equality (4.49) holds.

50) According to the equality (3.23), for every x∈(1,+∞),

$$(\mathrm{ch}_1^{-1}x)'=(\ln(x-\sqrt{x^2-1}))'=\frac{1-\frac{x}{\sqrt{x^2-1}}}{x-\sqrt{x^2-1}}=\frac{\frac{\sqrt{x^2-1}-x}{\sqrt{x^2-1}}}{x-\sqrt{x^2-1}}=-\frac{1}{\sqrt{x^2-1}};$$

so, the equality (4.50) holds. On the other hand, according to the equality (3.24), for every x∈(1,+∞),

$$(\mathrm{ch}_2^{-1}x)'=(\ln(x+\sqrt{x^2-1}))'=\frac{1+\frac{x}{\sqrt{x^2-1}}}{x+\sqrt{x^2-1}}=\frac{\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}}}{x+\sqrt{x^2-1}}=\frac{1}{\sqrt{x^2-1}};$$

which shows that the equality (4.50′) holds also.

51) According to the equality (3.25), for every x∈(-1,1),

$$(\mathrm{th}^{-1}x)'=\left(\frac{1}{2}\cdot\ln\frac{1+x}{1-x}\right)'=\frac{1}{2}\cdot\left(\ln\frac{1+x}{1-x}\right)'=\frac{1}{2}\cdot\frac{1-x}{1+x}\cdot\frac{1-x+1+x}{(1-x)^2}=\frac{1}{2}\cdot\frac{1-x}{1+x}\cdot\frac{2}{(1-x)^2}=\frac{1}{1-x^2};$$

so, the equality (4.51) holds.

52) According to the equalities (3.26), for every x∈(-∞,-1),

$$(\mathrm{cth}_1^{-1}x)'=\left(\frac{1}{2}\cdot\ln\frac{x+1}{x-1}\right)'=\frac{1}{2}\cdot\left(\ln\frac{x+1}{x-1}\right)'=\frac{1}{2}\cdot\frac{x-1}{x+1}\cdot\frac{x-1-x-1}{(x-1)^2}=\frac{1}{2}\cdot\frac{x-1}{1+x}\cdot\frac{-2}{(x-1)^2}$$

$$=\frac{1}{1-x^2};$$

so, the equality (4.52) holds. The other equalities - (4.52′) and (4.52″) - are obtained analogously.

53) According to the equality (3.27), for every x∈(0,1),

$$(\mathrm{sch}_1^{-1}(x))'=\left(\ln\left(\frac{1-\sqrt{1-x^2}}{x}\right)\right)'=\frac{x}{1-\sqrt{1-x^2}}\cdot\frac{\frac{x}{\sqrt{1-x^2}}\cdot x-1+\sqrt{1-x^2}}{x^2}$$

$$=\frac{1}{1-\sqrt{1-x^2}}\cdot\frac{x^2-\sqrt{1-x^2}+1-x^2}{x\cdot\sqrt{1-x^2}}=\frac{1}{1-\sqrt{1-x^2}}\cdot\frac{1-\sqrt{1-x^2}}{x\cdot\sqrt{1-x^2}}=\frac{1}{x\sqrt{1-x^2}};$$

so, the equality (4.53) holds. On the other hand, according to the equality (3.28), for every x∈(0,1),

$$(\mathrm{sch}_2^{-1}(x))'=\left(\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)\right)'=\frac{x}{1+\sqrt{1-x^2}}\cdot\frac{\frac{-x}{\sqrt{1-x^2}}\cdot x-1-\sqrt{1-x^2}}{x^2}$$

$$=\frac{1}{1+\sqrt{1-x^2}}\cdot\frac{-x^2-\sqrt{1-x^2}-1+x^2}{x\cdot\sqrt{1-x^2}}=\frac{1}{1+\sqrt{1-x^2}}\cdot\frac{-1-\sqrt{1-x^2}}{x\cdot\sqrt{1-x^2}}=-\frac{1}{x\sqrt{1-x^2}};$$

which shows that the equality (4.53′) holds also.

54) According to the equality (3.29), for every x∈(-∞,0),

$$(\mathrm{csh}_1^{-1}(x))'=\left(\ln\left(\frac{1-\sqrt{1+x^2}}{x}\right)\right)'=\frac{x}{1-\sqrt{1+x^2}}\cdot\frac{\frac{-x}{\sqrt{1+x^2}}\cdot x-1+\sqrt{1+x^2}}{x^2}$$

$$=\frac{1}{1-\sqrt{1+x^2}}\cdot\frac{-x^2-\sqrt{1+x^2}+1+x^2}{x\cdot\sqrt{1+x^2}}=\frac{1}{1-\sqrt{1+x^2}}\cdot\frac{1-\sqrt{1+x^2}}{x\cdot\sqrt{1+x^2}}=\frac{1}{x\cdot\sqrt{x^2+1}};$$

so, the equality (4.54) holds. On the other hand, according to the equality (2.30), for every x∈(0,+∞),

$$(\mathrm{csh}_2^{-1}(x))' = \left(\ln\left(\frac{1+\sqrt{1+x^2}}{x}\right)\right)' = \frac{x}{1+\sqrt{1+x^2}} \cdot \frac{\frac{x}{\sqrt{1+x^2}} \cdot x - 1 - \sqrt{1+x^2}}{x^2}$$

$$= \frac{1}{1+\sqrt{1+x^2}} \cdot \frac{x^2 - \sqrt{1+x^2} - 1 - x^2}{x \cdot \sqrt{1+x^2}} = \frac{1}{1+\sqrt{1-x^2}} \cdot \frac{-1-\sqrt{1+x^2}}{x \cdot \sqrt{1+x^2}} = -\frac{1}{x \cdot \sqrt{x^2+1}};$$

which shows that the equality (4.54′) holds also.

5. Conclusions

As you can see, in this paper we presented 54 other properties of the hyperbolic functions, divided into three groups. And here, as in (Vălcan, 2016), the aim was to form the reader's attention and interest in these issues, developing their global image about these features and their inversions. Precisely why the demonstrations are presented in full, in detail, so that it can be used in the classroom.

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