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On A Rectangular Hyperbola and Two Conjugate Sheaves of Circles ("2nd Order Curves On a Soccer field")

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Abstract

This article should be considered in line with the so-called problem-based learning, which has been developed by the American psychologist and educator John Dewey (1859 -1952). This method requires the creation in the learning process of problem situations that need to meet the objectives of the education system and at the same time be accessible to students and stimulate their own active learning activities. Most often the problematic situation is detected in the process of solving nonstandard problems. Nowadays, this topic in pedagogy is receiving more attention. Many papers have been published on this topic. We can cite among others, (Махмутов 1977), (Azer 2011, 808), (Barrows 1996, 3) and (Wood 2003, 328). Our work illustrates in fact this type of problem situations in a rather peculiar example, which shows how you can consistently direct the learning process.

Introduction

(Azer 2011, 808) presents the difficulties that an institution might face while trying to implement this approach. Among other things, he cites the need to prepare faculty for the change, establish a new curriculum committee and working group that could prepare the program in designing the new problem-based learning curriculum and redefine educational outcomes. A brief overview of the method is provided in (Barrows 1999, 3), while (Wood 2003, 328) argues on the importance and impact of this method in medical education. We argue that even though it is hard to implement this method in general for an institution, it has shown to be very useful for individual instructors. With enough courage and dedication, an instructor can get his students to understand better the subject in the setup of a classroom. In running an undergraduate seminar, this method brings more excitement to students and they often prove to be more knowledgeable than their peers.

In this paper, we try to apply this method to the unusual integration of topics from mathematics that students often learn as separate unrelated atoms. We propose to the teacher to put a simple task, arising from a game situation well understood by the students. In the process of solving the problem, we shall modify the problem after every solution linking the new targets to the outcomes of the previous questions, making them more complicated. This is intended to encourage students at each step to build up the intellectual capacity to understand the new formulation of the problem and search for ways to solve it. At each step, students feel the need to boost their knowledge of different branches of mathematics that are necessary for a better understanding of the challenges and opportunities for their development using new ideas. On the example of a soccer game (In fact, it can be any game, in which points are scored for hitting a target from a rectangular area), we take a student from the basic concepts of analytic geometry, beyond the standard course and encourage the students to a deeper study of the interesting and nontrivial topic of geometry as sheaves of circles, which could serve as the theme of a course work and even encourage youngsters to independently obtain new results.

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Problem-based learning is remarkable in that it calls for an intellectual motivation of students, aimed at enhancing their thinking skills and ability to independently solve problems. The key to this technique is the creation of the so-called problem situation, to which various approaches can be applied. The problem at the origin of this paper is a typical example of such a situation. In some sense, we confirm Hilbert's famous quote: "Mathematical science is in my opinion an indivisible whole, an organism whose vitality is conditioned upon the connection of its parts."(Лонгла 2003) presented a soccer problem at a conference in Moscow. The problem was formulated as follows. "Children are playing soccer. Moving with the ball towards the goal line of the opposite side parallel to the sideline, the striker sees the target at a variable angle α. To have a shot on target, it is necessary to determine at what distance x from the goal line of size 2a visibility is at the maximal angle α_0."This problem can be used to introduce the student to various aspects of mathematics, starting from high school geometry. This is what we do in the subsequent sections.

I. **A High school geometry approach**

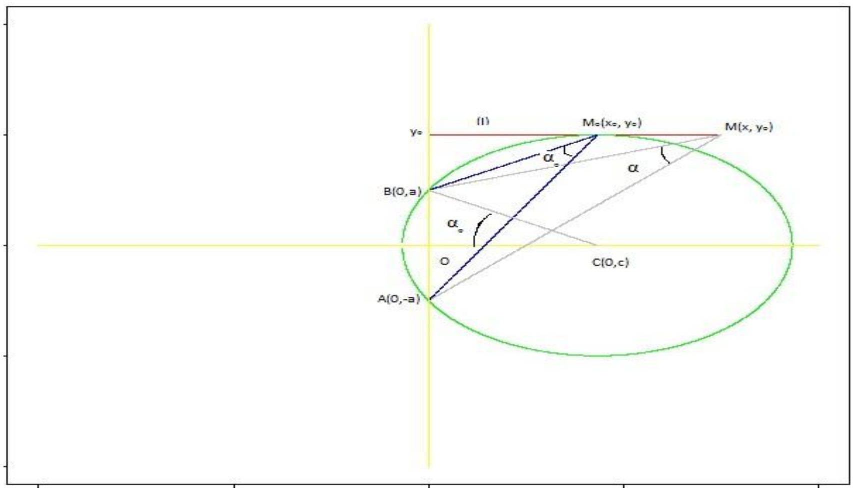

Figure 1: Geometric solution.

Using notations from Figure 1, we can derive the following. From any position $M(x,y_0)$, draw a line (l) that is orthogonal to the goal line. Draw a circle that is tangent to this line and passes through the end points of the goal A and B. The intersection of the circle and this line is $M_0(x_0,y_0)$. This point is independent of x. We argue that this point is the point from which the player sees the goal with a maximal angle.

It is well known that the angle inscribed in a circle, and based on a chord is larger than the angle with sides passing through the end points of the chord, if the corner point of this angle lies outside the circle. So, the angle at M_0 is by construction larger than that of any other point on the line (l). Also, this angle (AM_0B) is equal to half of the central angle subtended to the chord (ACB). Therefore, the solution of the problem is the intersection of the circle and the line on which the player moves, when y = b with b> a.

Figure 1 shows the corresponding coordinate system (OXY). The center of the circle is at the point C(c, 0), where $c = \sqrt{b^2 - a^2}$. The maximal angle α_0 is determined by the formula: $\tan\alpha_0 = \frac{a}{\sqrt{b^2-a^2}}$.

II. **A calculus approach**

This problem can also be used to integrate trigonometry and calculus. In fact, after introducing the notion of derivative and properties, one can propose this example to show to the students that mathematical topics need to be integrated, combined and understood as parts of a whole. We would say, let β_1, β_2 be respectively the angles formed by BM and AM with the line of motion. The student knows that $\tan\alpha = \tan(\beta_2 - \beta_1) = \frac{\tan\beta_2 - \tan\beta_1}{1+\tan\beta_1\tan\beta_2}$,

$\tan\beta_2 = \frac{b+a}{x}$ and $\tan\beta_1 = \frac{b-a}{x}$. Thus, he can derive the function that needs to be maximized. This function needs to be maximized because we are looking for the largest angle and it is clear that for positive angles smaller than the right angle, the tangent function is increasing. We have $f(x) = \tan\alpha = \frac{2ax}{x^2 + b^2 - a^2}$. The student can then easily take the derivatives and find the solution $x = \sqrt{b^2 - a^2}$, for which the maximal angle is again found via $\tan\alpha = \frac{a}{\sqrt{b^2 - a^2}}$, using the second derivative test or the first derivative test. It is at this point recommended to comment on the above two solutions, to show the student how connected are various parts of mathematics. Geometric arguments did not require any knowledge of calculus, and the arguments use in the calculus approach make no use of the geometric approach to the solution. But in the end, the unique solution to the problem is once more derived.

III. **The locus of these points is a hyperbola.**

Continuing the study of this game, we can ask the student to identify the set of all such points, when the line of motion varies. Arguing abstractly that the game could be played on both sides of the target, along each line we will find two points. It is clear that the x component of the center of the circle changes with b by the formula $c = \sqrt{b^2 - a^2}$. Thus, all points of interest satisfy the equation $y^2 - x^2 = a^2$ - rectangular hyperbola. We obtain here a rectangular hyperbola as set of points from which a given target is seen under a maximal angle for viewers on a line orthogonal to the target line, when not allowed to be in the corridor based on the target. See Figure 2 below.

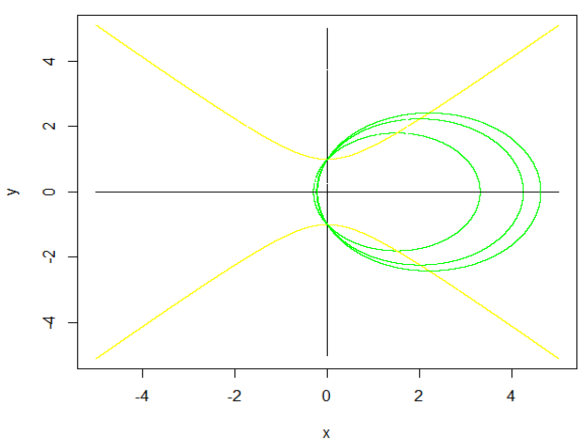

Figure 2: Hyperbola.

IV. **Two sheaves of circles.**

Recall that the geometric interpretation of this problem involved a circle centered at C(c,0). Let $L_c: x^2 + y^2 - 2cx - a^2 = 0$ denote the equation of this circle. Notice that any circle of this family passes through 2 fixed points $A(0, -a)$ and $B(0, a)$. Such a family is called an elliptic sheaf of circles. Now, consider a line $y = b$ with b>a, that crosses a circle of the sheaf at two points $C_1(c_1, b)$ and $C_2(c_2, b)$. It is clear that c_1, c_2 are solutions of the equation $x^2 - 2cx + b^2 - a^2 = 0$, and $c_1c_2 = b^2 - a^2 = (b-a)(b+a)$. This is the product of the secant drawn from the point D (0, b) to this circle (DA) and its outer part (DB).

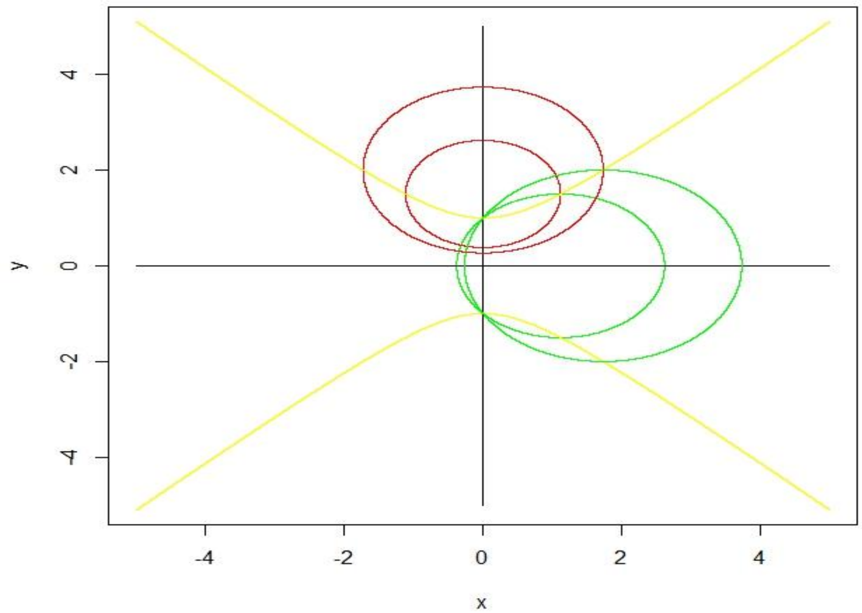

Figure 3: Hyperbola and circles.

It is easy to show that the product c_1c_2 is equal to the square of the length of the tangent segment to the circle drawn from the given point. Since this product is independent of c, then for a fixed value of b, it is going to remain constant (including the case of complex conjugate roots) for all circles of the sheaf. Therefore, all line segments from D(0,b) that are tangent to these circles have the same length $\sqrt{b^2 - a^2}$ and can be considered as radii of a circle centered at this point. The equation of this circle is $L_b: x^2 + y^2 - 2by + a^2 = 0$. All circles of the family L_b intersect at two purely imaginary points $(-ai, 0)$ and $(ai, 0)$. Therefore, the sheaf will be called hyperbolic. Notice that the points $A(0, -a)$, $B(0, a)$, $A_1(-ai, 0)$ and $A_2(ai, 0)$ are nothing but the real and imaginary vertices of our initial hyperbola.

The goal is now to show that circles from these two sheaves are tangent (see Figure 3). At this point, one can use various approaches to get the students through other topics and areas of mathematics. It would particularly be recommended to use this part in group problem sessions or for an undergraduate seminar for mathematics majors. The following section can of course be used in other circumstances to help students from other majors in integrating their mathematics abilities. We will say that a circle from L_b is conjugate to a circle from L_c if and only if $c^2+a^2=b^2$. On Figure 3 students can clearly see that circles from L_c have two fixed points, circles from L_b don't intersect and the parabola passes through the intersection of conjugate circles. In fact, the set of all intersections of conjugate circles form the parabola.

V. **The analytic approach to the proof of orthogonality**

We need to show that any circle from L_b is orthogonal to its counterpart from L_c. This means that we want to show that the tangent lines to these circles at their point of intersection are perpendicular. In other words, we will show that the product of the slopes of the tangent lines at the point of intersection of these circles is equal to -1.

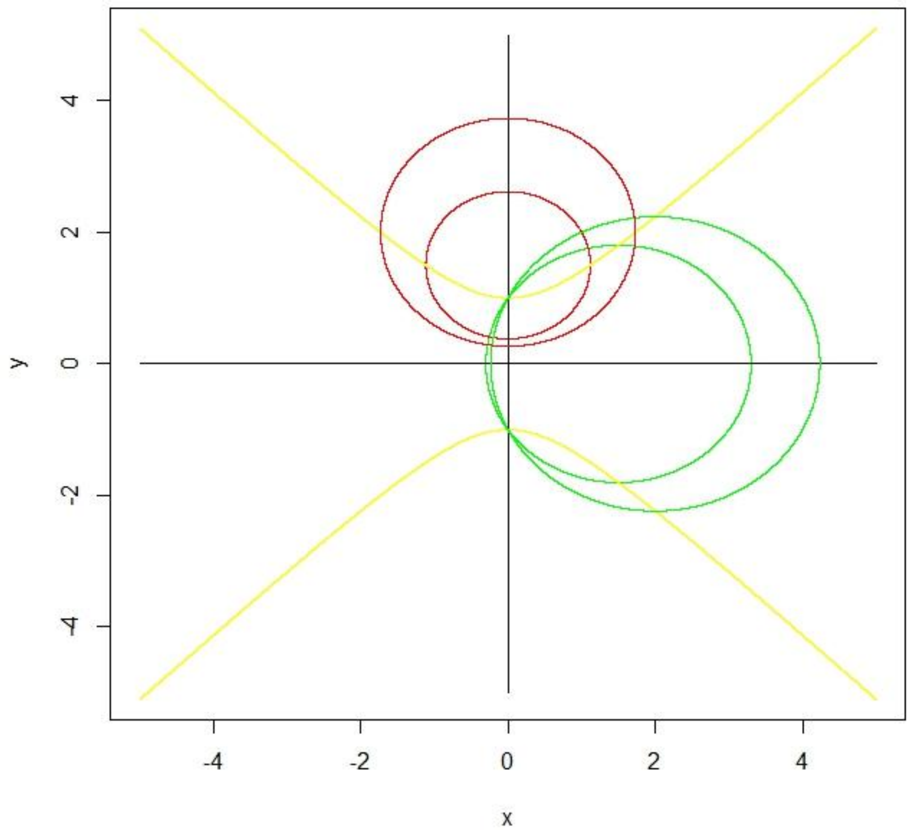

Figure 4: Orthogonal circles

Considering the problem in the setup of calculus, we could use implicit differentiation to show that the two families are orthogonal. In fact, differentiating both sides of the two equations, we have

$$L_c: 2xdx - 2cydy + 2ydy = 0, \; L_b: 2ydy - 2bdy + 2xdx = 0.$$

The student easily finds the slope of the respective tangent lines to $m_c = \frac{c-x}{y}$ and $m_b = \frac{x}{b-y}$. Notice that at the intersection of the two circles, $m_c . m_b = \frac{xc-x^2}{yb-y^2} = -1$. The last equality can be obtained by adding side by side the equations of the two circles at their intersection. This shows that the tangent lines to the circles are orthogonal at their intersection. Therefore, the circles are orthogonal. See Figure 4.

VI. **Ordinary differential equations' approach to orthogonality**

Now, we turn to the use of some notions of differential equations on this problem. Consider that these circles are integral curves of some ordinary differential equations. This can be done because each of the families is a one parameter family of curves, from each of which the parameter can be excluded. The student has the task to find these ordinary differential equations and check the fact that their respective solutions are orthogonal. This is also done by showing as in the previous method that the product of the slopes of the respective tangent lines is -1 at the point of intersection. It is obvious that to find the ordinary differential equation that the integral curve solves, one needs to differentiate their form and solve for the constant to exclude it from the equation of the curves. In doing this, we obtain: $L_c: x^2 - 2cx + y^2 - a^2 = 0$, which gives $2x - 2c + 2yy_c' = 0$ and $L_b: x^2 - 2by + y^2 + a^2 = 0$, leading to $2x - 2by_b' + 2yy_b' = 0$. Thus, solving for the parameters and substituting in the formulas of the curves gives $y_c' = \frac{-x^2 + y^2 - a^2}{2xy}$ and $y_b' = \frac{2xy}{x^2 - y^2 + a^2}$.. Therefore, once more, we get to $y_c' . y_b' = -1$. This shows once more, that all correct methods lead to the same solution in mathematics. Notice that for points of our original hyperbola $y^2 - x^2 = a^2$, the tangent lines to circles from L_b are vertical (slope infinity) and the tangent lines to circles from L_c are horizontal (slope 0). In other words, the far right points of circles from L_b are the top points of those of L_c. See Figure 3.

VII.**Radical axes and conjugacy of the sheaves.**

Geometry of circles studies the concept of power of a point. See (Postnikov 1973). Recall that one of the equivalent definitions of the power of a point is the following. If a point is outside the circle, its power is equal to the square of the length of the tangent segment drawn from it to the circle. As reported earlier, it is also equal to the product of the length of any secant drawn from the given point and its outer part. Considering a couple of circles, the set of all points for which the powers relative to each of the two circles are equal to each other is called radical axis of these circles. Looking back at IV, we see that the line of centers D (0, b) of circles from the family L_b has this property relative to L_c by construction. Thus, the OY axis is a radical line of all circles of the sheaf L_c. We obtained this by considering various secants to circles of the sheaf L_b, parallel to the OX axis (parallel to the line of centers of circles from L_c). Now, the student can show that the line of centers of circles from the sheaf L_c is a radical axis of circles from the sheaf L_b. In fact, considering the line of centers of two circles from L_b and L_c as a secant $(\sqrt{b^2 + c^2} + \sqrt{b^2 - a^2})$ drawn from the point C (c, 0) to the circle of radius $\sqrt{b^2 - a^2}$ of L_b, its product with its outer part $(\sqrt{b^2 + c^2} - \sqrt{b^2 - a^2})$ is $c^2 + a^2$ - independent of b. Consequently, the order of the point C relative to all circles of L_b is same. In view of the arbitrary choice of the point C, this means that the line of centers of circles of L_c (OX axis) is a radical axis of circles from L_b. Thus, it has been shown that the line of centers of circles from each of the sheaves is a radical axis of circles from the other sheaf. Such sheaves of circles are called conjugate sheaves.

Thus, at the referee's whistle for the end of the game, we can jokingly say that football is played not only with the body, but also with the brain. Therefore, stealing from Blaise Pascal, we can say that "with equal minds, wins the one who knows geometry!"

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