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Updating, Modernizing, and Testing Polya's Theory of [Mathematical] Problem Solving in Terms of Current Cognitive, Affective, and Information Processing Theories of Learning, Emotions, and Complex Performances.

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Abstract

Polya's classic work *How to Solve It: a New Aspect of Mathematical Method* still ranks high on various lists of most read and most referenced books. Polya claims that true problem solving is accompanied by the cognitive activities of mobilization, organization, isolation and combination, and by the meta-cognitive evaluations of relevancy, proximity, and quality. These meta-cognitive evaluations occur as a result of monitoring cognitive activities on an on-going basis. According to Polya, these particular meta-cognitive activities are a necessary part of true [mathematical] problem solving and they generate positive, negative and oscillating emotions during problem solving which help or hinder obtaining a solution to the problem. The studies we have done, which are summarized in this article, have provided good initial and formative support of Polya's model and particularly so in terms of the disruptive influence of emotions (both positive and negative) and the oscillation of emotions during mathematical problem solving and particularly for difficult mathematical problems, which we call "Polya Problems." As part of our inquiry, we updated and modernized Polya's theory in terms of research that has occurred and understanding that have been developed in psychology in the last twenty years. Both Polya's theory and our updates and modernizations of it are presented in detail.

Keywords: Mathematical problem solving, emotions, emotions and problem solving, complex problem solving, meta-cognition, self-regulation, cognitive development.

1. Introduction

Given the current national emphasis on problem solving in mathematics, we believe that is extremely important to have a validated set of mathematical problems that operationalize a well-formulated theory and view of *what a mathematical problem is* (as well as what constitutes a solution) in order to study mathematical problem solving systematically and programmatically and in a manner that makes results easily comparable from study to study. Further, we believe that Polya (1945, 1985) has one of the best-formulated theories and views of what a problem is and what mathematical problem solving entails, particularly when it is updated and modernized in terms of research that has occurred and understanding that have been developed in psychology in the last twenty years. We are not alone in this view of Polya and his work on mathematical problem solving, which is also a general theory of problem solving as well (see Schoenfeld, 2000 and Cobb, 2006 for details), even though this view is still 'emerging' as a mainstream view of problem solving in psychology and education.

Polya's classic work *How to Solve It: a New Aspect of Mathematical Method* ranks close to Kuhn's *Structure of Scientific Revolutions* in number of printings, copies sold, and impact, although Polya has not been as much read in the last decade as previous ones, particularly in the area of mathematics education and psychology.

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Polya has very specific views and criteria as to what constitutes a mathematical problem and why, and it is for this reason that we have coined the term "Polya Problem" to distinguish such problems from other kinds of problems that are typically used in studies of mathematical and non-mathematical problems solving. The majority of the problems used in many studies of mathematical problem solving would not meet Polya's definition and criteria of what a mathematical problem is (see Higbee & Thomas, 1999). This same point is true for non-mathematical (convergent) problem solving studies as well. The exception to this latter point, however, is work that has been done on fuzzy, ill-defined problems and problem solving where there are multiple partial or imperfect solutions, which is a class of problem and problem solving not typically associated with common and convergent as opposed to more "exotic" mathematics. It is for these reasons, then, that we chose to develop and validate a set of 24 "Polya Problems," which constitute three parallel subsets of problems so that fresh problems may be used in repeated testing situations (see Allen and Carifio, 2005 for details). Two of these problems, one comparatively easy and one much harder, were randomly chosen from this problem set in the studies we have done to test various aspects and claims of Polya's theory and views of mathematical problem solving.

2.0 Cognitive Processes

According to Polya (1981, 1985), there are certain key cognitive activities that occur during problem solving. These activities are *mobilization, organization, isolation, and combination.* During true problem solving, once a problem solver comes to understand the statement or nature of a problem (which entails analysis and comprehension in Bloom's parlance), various pieces of information relevant to the problem must be recalled from memory. Polya calls the process of retrieving relevant elements from memory, such as solved problems, theorems, and definitions, *mobilization,* which implies that the process is something more than mere (automatic and mechanical) recall and more akin to goal directed meta-cognitive memory searching and knowledge review. Next, initially unconnected facts are grouped together and adapted to the problem. Polya calls this step the process of constructing *connections* (i.e., stipulated relationships or propositions) between recalled information and key problem elements, and developing the problem's *organization* (or schema/representation in modern parlance). The way in which information from the problem and information retrieved from memory is organized into a synthetic whole (problem schema or gestalt) is critical for remembering and understanding various aspects of the problem (Mandler, 1984). A plethora and wide variety of research in cognitive psychology over the past twenty years has strongly confirmed these later points to be the case (see Ashcraft, 2002 for details). Also, the process of developing an accurate and effective problem organization is a highly active process (DeBellis & Goldin, 1997), which requires convergent synthesis (see Guilford, 1958) and meta-cognitive skills and abilities (see Meyer, 1998; and Dinsmore, Alexander, & Loughlin, 2008) as well as self-monitoring and self-regulation skills (see Thiede, Anderson, & Therriault, 2003), as do many other problem solving sub-skills and processes. For example, problem solvers who are low or shallow elaborators tend to produce less accurate and effective organizations than problem solvers who are high or extensive elaborators (Goldin, 2000). Elaboration, however, is a meta-cognitive process, strategy, and skills set that can be taught and developed over time and "habituated" (see Ashcraft, 2002).

The process of focusing on a single detail (or element) and isolating it from the rest of the problem is called *isolation* by Polya (or disembedding in newer views and models). The problem solver must single out and isolate specific details and their relevance and usefulness and connections to other elements in solving the problem. Isolation of key elements and details often is difficult for problem solvers who tend to be field dependent as opposed to field independent (see Akins, 1987 for details), and teaching this process and skills set to field dependent subjects is often quite difficult, and they often must learn several alternative compensating strategies relative to this key process (Nasser &Carifio, 1993).

Polya calls grouping details together in a new way *combination* (which is very similar to divergent synthesis in Guildford's Structure of the Intellect model of cognitive processes, 1958). By viewing the problem, or parts of the problem from different perspectives and from different sub-perspectives, the total conception of the problem can be improved and *iteratively* bring the problem solver closer to a solution. Implied in Polya's model is the view that problems *are solved in stages and steps along solution paths* that may be productive, progressive, and energizing, or tortuous, dead-ended, frustrating and even demoralizing. Polya's key and critical point here is that solution stages and solution paths have *positive or negative outcomes to some degree,* and thus they do not follow one another in a straight line necessarily to a solution.

For Polya, therefore, solving a true problem and true problem solving tends most often to be circuitous and non-linear and rift with disappointments and failures as well as elations and success, which is a very different view from other problem solving models (e.g., Thorndike, 1911; Gagne, 1985; and Torpe and Sage, 2002) in a number of different ways. It is also for these and other reasons that the problem solver's monitoring and evaluating of her or his progress towards a solution, and revising behaviors and strategies accordingly based on this real time data, is such a key meta-cognitive process and skills set for Polya, which he has written about extensively.

3.0 Meta-cognitive Processes and Emotions

There are on-going *meta-cognitive evaluations* and *resulting emotions* that occur during true problems solving according to Polya. Further, there are three general kinds of meta-cognitive evaluations a problem-solver makes on an ongoing basis in Polya's view. These are relevancy, proximity, and quality. *Relevancy* is how important solving the problem is to the problem solver, or how important a particular idea or guess is in obtaining a solution. *Proximity* is the problem solver's estimate of how close or far away she or he is from a solution. Described by Polya, but not named as such, *Quality* is an estimate of how good an idea or guess is, or how well the problem solver believes the problem solving process is proceeding for her or him.

Polya contends that the results of these on-going meta-cognitive evaluations of relevancy, proximity, and quality bring about *emotions* in the problem solving process, with the meta-cognitive appraisals leading to physiological emotional responses (an implicit view in Polya's writings). This view is consonant with Mandler's (1989) well-known view in contemporary psychology. Mandler's view contends that emotions arise from *interruptions* in thoughts. An on-going (meta-cognitive) evaluation process interprets an interruption as positive or negative depending on the *incongruity* between what the problem solver is expecting and the problem solver's perception of the results provided to her or him by the actual event. This process and event is called *cognitive dissonance* or *cognitive confirmation* in other somewhat similar contemporary views. Mandler (1989) has shown that that change in physiological activity (levels) is the direct consequence of cognitive incongruity and interruption or cognitive confirmation. Mandler argues that *emotional tone* (whether agreeable or disagreeable) depends on cognitive evaluations, whereas *emotional intensity* depends on the degree of physiological activity. It should be noted, however, that physiological responses and activities may also have their own associated *parallel arousal processes* (from "bubble" and/or episodic as opposed to semantic long term memory), which, *if unregulated and unmanaged,* may completely disrupt and/or abruptly shut down the problem solving process due to their intensity, creating the well-noted states of fear, anger, despair, and demoralization observed in many problem solvers working on "hard' or "difficult" problems *to and for them.*

This notion and reasonably supported view of Mandler's that doing mathematics and mathematical problem solving can actually "hurt" (*literally and physically*) to the point of being highly to completely disruptive while doing mathematical problem solving is also supported in part by a recent fMRI research study done by Lyons and Beilock (2012) that showed the higher a person's *anxiety* about mathematics and doing mathematics is, the higher the activation of the posterior insula, which is a fold of tissue located deep in the brain just above the ear that is associated with registering direct threats to the body as well as the experience of pain. This finding was both strong and clear in Lyons and Beilock's work and in prior work showing that the phenomenon and effect was present in third graders (Beilock, 2008). However, the critical experimental flaw in Lyons and Beliock's work is that they only gave *adult* subjects extremely simple and **rote** math problems to do (e.g., [12x4]-19) and not Polya problems, such as the one given in Table 2, which is why Lyons and Beliock did not observe any increases in subjects' physical pain levels while working "math problems." This critical flaw is a very common experimental flaw in mathematics and mathematical problem solving research, particularly in those studies done outside of the mathematics education area. If Lyons and Beliock had used Polya problems, and particularly as a contract to the extremely simple and rote math problems they used, they most certainly would have seen the cognitive and affective disruptions and increased pain levels predicted by both Mandler and Polya during the problem solving processes of these adult subjects, as well as the lowering of anxiety and pain levels as they were making progress on devising solutions to the Polya problem, and particularly so when they solved it. It is this *oscillation* of anxiety and pain (and joy and pleasure) during real [mathematical] problem solving, and the control and management of *both,* as well as the using of *both* to monitor ongoing progress and lack of progress information for decision-making that is one of the most *distinctive* aspects of Polya's theory.

It is this particular real time and oscillating emotions (as a result of cognitive and meta-cognitive engagement processes) view of mathematical problem solving that makes Polya's views still relevant and different from most views in the area of [mathematical] problem solving today, which tend not to be dynamic process models with strong sematic, cognitive, meta-cognitive, affective and oscillating processes components.

Table 1 presents a summary of the key cognitive and meta-cognitive processes and components of Polya's updated and modernized theory of mathematical problem solving. As may be observed from Table 1, to be a true problem to Polya, the problem must have characteristics which engage the problem-solvers emotions as well as intellect, and the problem-solver will have positive as well as negative emotions during the process of solving the problem, which tends not to occur while doing rote problems, problems done rotely, or exercises. Further, *regulating* one's emotions and meta-cognitive processes is an important part of the problem-solving process in Polya's view of problem solving. According to Polya, one must develop these meta-cognitive and self-regulating skills and processes in the problem solver to have a good problem-solver and good problem solving of real problems, as poor problem-solving abilities and failures to solve problems can result from poor meta-cognitive and self-management skills (and the other factors discussed above), as well as the resulting negative emotions and unwillingness to make fresh attempts at both current and future problems. For Polya, one must attend to and manage the cognitive, emotional, and meta-cognitive components of problems and problem-solving to both understand and develop problem-solvers who have the skills and abilities to solve real problems twenty years in the future that, due to progress, will more than likely be unlike those done today, or when the problem-solver was being formally schooled.

Table 1: The Cognitive and Meta-Cognitive Processes and Components of Polya's Updated and Modernized Theory of Mathematical Problem Solving.

Basic Cognitive Operations (Performed Iteratively in Any Sequence)

Isolation

Key elements of the problems are disembedded and isolated out from other elements;

Key elements are broken down into narrower parts; details are separated and made more distinct ; a single detail is isolated, concentrated on, and studied.

Mobilization

Key elements of the problem are more fully and deeply recognized and processed; relevant information about key elements is recalled from long term memory**.**

Organization

Unconnected facts and elements are grouped together and adapted to the problem; stipulated connections between recalled elements and key problem elements made; details are added from memory or through elaboration to supplement the problem; elements of a part of the problem are rearranged or grouped in a new way; a part of the problem formerly in the foreground recedes into the background; a problem schema is formulated either explicitly or implicitly.

Combination

Problem elements and sub-element are viewed from different perspectives; all the elements details are combined in a new, more harmonious way; the whole problem is viewed in a new way; a new problem schema is formed through divergent synthesis (Aaah Ha!).

Basic Meta Cognitive Operations (Performed Iteratively in Any Sequence) which Generate Emotions in the Problem-Solver.

Relevancy

How important solving the problem is to the problem solver;

How important a particularly idea of guess is in obtaining a solution.

Proximity

The problem-solver's estimate of how close or far she or her is from a solution

Quality

Estimate of how good an idea or guess is; how well the problem-solver believes the problem solving process is proceeding for her or him.

4.0 Characteristics of Polya Problems

Given the above, a problem to Polya and thus a "Polya Problem" must be (1) comprised of several elements that need to be related together (some of which may or may not be relevant), (2) require several steps to get a solution, (3) have several different potential solution paths and (4) require information to be furnished from outside of the problem statement to produce a solution. A "problem" that had one or two elements that required one step to solve and contained all, or almost all, of the information needed to obtain a solution in the problem statement would not be much of a "problem" to Polya or a "real (Polya) problem," but more on an exercise (see below for details) that could be done rotely and done without understanding. Doing simple arithmetic would not be problem solving to Polya, not only because this performance could be nothing more than a rotely executed "habit" or algorithm, but also because *Polya's model implicitly assumes that the problem solver is at the upper concrete operations level to formal reasoning level of cognitive development.*

Polya's model of problems and problem solving is **not** about the problem solving of pre-formal or early concrete operations children, although many of the proto-elements of his model may be both observed in and taught to such children to better prepare them for their next developmental levels. Polya's model is a model of problems and problem solving of *young adult to adult problem solvers,* which is an extremely important point about Polya and his model, as the two types of problem solvers are extremely and qualitatively different from each other. Polya's model is not about elementary problems and problem solving, but rather it is about more sophisticated problems and problem solving that are more complex and "real world" in character and the level of the cognitive development of the problem solver is a critical and limiting factor in his model and with using it.

Polya considers **mobilization** and **organization** to be complementary activities in the way they work together. Similarly, **isolation** and **combination** are complementary activities. In proceeding towards a solution, the operations of mobilization, organization, isolation and combination may occur repeatedly and in any sequence. Progress may be slow and may take place in imperceptible steps. If the steps lead down a dead end path, the problem solver must repeat the mobilization and organization operations before further isolation and combination activities can occur. Understanding that one is recognizing and eliminating unproductive solutions paths which in itself is a positive step along the path and process towards a solution, particularly if one is doing this reasonably quickly, would be the meta-cognition of an "optimistic" (and continuing to preserve) as opposed to a "pessimistic" (and possibly quitting) problem solver. Encouraging, teaching, and rewarding "*problem solving optimism*" would be the preferred practice in Polya's model rather than the opposite. Also, Polya is quite clear about failure in problem solving containing helpful and potentially positive information relative to solving the problem and being a step along the path to problem solving success and not a signal for terminating further attempts to solve the problem or attempts at future such problems, which is often the "teaching message" in many areas and approaches to teaching mathematics, problem solving and mathematical problem solving. Polya advocated *post-mortems* of problem solving "failures" to learn from the failures and find the helpful and potentially positive information relative to solving the problem.

Polya diagrams the activities that occur during problem solving by arranging his four operations on opposite corners of a square. The edges of the square represent the cognitive activity sequences that are likely to occur repeatedly while solving the problem.

These sequences include the operations of recognizing details, remembering relevant aspects of the problem, supplementing the problem with more information, and combining the information in new ways. Each sequence of activities should lead to an improved conception of the problem. If the problem is a "Polya Problem" and requires a number of steps to solve that the problem solver must detail in her or his answer, the various steps indicated by the problem solver should be able to be categorized in terms of Polya's four basic problem-solving functions. Polya contends that solving a mathematic problem requires all four of the functions he specifies and that a problem that does not require the four activities in his model is not a real problem but only an *exercise* (or rote or algorithmic application). The difference between a real problem or a "Polya Problem" and a rote or algorithmic exercise, therefore, is an important and critical distinction in Polya's view, and the process or task of completing an exercise is not problem solving according to Polya.

5.0 Polya Problems

As previously stated, we developed and validated three (3) parallel problem sets of 8 problems in each set that ranged from "easier to "harder" that met the various criteria state above for a Polya problem in order to allow various aspects Polya's theory of mathematical problem solving outlined above to be observed and tested. The reader is referred to Allen and Carifio (2005) for details on the specifics, development and validation of these three Polya problem sets.

An example of a Polya problem and model Polya answer with the solution steps labeled in terms of Polya's four key cognitive processes (mobilization, organization, isolation, and combination) is given in Table 2 to show the reader the concrete details of a Polya problem and a model response to the problem sated. As can be seen from Table 2, it takes 14 steps to produce a model answer to this Polya problem, which cognitively entails in engaging in *isolation* processes 3 times, *organizational* process 6 times, *mobilization* processes 3 times and *combination* processes 2 times. As can also be seen from Table 2, one would be hard pressed to call this Polya problem a problem that could be solved rotely or algorithmically.

Table 2: An Example of a Polya Problem and Model Polya Answer with the solution steps labeled in terms of Polya's four Key Cognitive Processes (mobilization, organization, isolation, and combination).

Example Polya Problem: Two spheres with one inch radii are situated so that each passes through the center of the other. Find the length of the curve of their intersection.

Steps in Developing a Solution:

Spheres, centers recognized (Mobilization)
intersection is a circle (Combination)
what is radius? (Isolation)
draw 2-D picture (Organization)
sides are radii of the spheres (Organization)
it is an equilateral triangle (Combination)
what is the height of the triangle? (Isolation)
what are angles of equilateral triangle? (Isolation)
look at half triangle (Organization)
remember 30-60-90 angle ratio => 1-2 root3 sides (Mobilization)
scale sides (Organization)
radius of circle =root3 /2 (Organization)
circumference = 2 pie (Mobilization)
apply: circumference = pie / root3 (Combination)
total number of steps..14

Frequency of Polya's Four Key Cognitive Processes in obtaining the above solution:

3 Isolation processes (steps)
6 Organization processes (steps)
3 Mobilization processes (steps)
2 Combination processes (steps)

Table 3 presents the holistic rubric for scoring students' responses to Polya problems that we have used in all of the studies of Polya problems we have done. This scoring rubric was developed and validated by the California State Department of Education Assessment Program (Meier, 1992) for evaluating student responses to mathematics Problems that entailed mathematic thinking and reasoning.

The California rubric development group recommends first sorting students' responses into three macro groups; good (5 or 6 points), adequate (3 or 4 points), or inadequate

Table 3: Holistic Rubric for Scoring Students' responses to Polya Problems

Exemplary response; score = 6.

The response is complete and includes a clear and accurate explanation of the techniques used to solve the problem. It includes accurate diagrams (where appropriate), identifies important information, shows full understanding of ideas and mathematical processes used in the solution, and clearly communicates this knowledge.

Competent response; score = 5.

This response is fairly complete and includes a reasonably clear explanation of the ideas and processes used. Solid supporting arguments are presented, but some aspect may not be as clearly or completely explained as possible.

Satisfactory with minor flaws; score = 4.

The problem is completed satisfactorily, but explanation is lacking in clarity or supporting evidence. The underlying mathematical principles are generally understood, but the diagram or description is inappropriate or unclear.

Nearly satisfactory, but contains serious flaws; score = 3.

The response is incomplete. The problem is either incomplete or major portions have been omitted. Major computational errors may exist, or a misuse of formulas or terms may be present. The response generally does not show full understanding of the mathematical concepts involved.

Begins problem but fails to complete solution; score = 2.

The response is incomplete and shows little or no understanding of the mathematical processes involved. Diagram or explanation is unclear.

Fails to begin effectively; score = 1.

The problem is not effectively represented. Parts of the problem may be copied, but no solution was attempted. Pertinent information was not identified.

No attempt at solution; score = 0.

No attempt at copying or solving the problem is made.

(0, 1, or 2 points). Each of these groups is then sorted into the categories in the subgroup. We employed this procedure in scoring these problems with the California rubric in all studies we have done of Polya problems to date.

6.0 Testing Polya's Problem Solving Model

The subjects in our exploratory study attended a public university in the Northeast. The first and third quartile SAT/ACT scores for students attending the university was 950 and 1150 respectively. Further, 17% were minorities, 13% were from other states, 3% were international, and 18% were more than 25 years old.

One hundred and fifty two (152) volunteer freshman and sophomore undergraduates were subjects in our pilot study. As this was a scientific-technical university, most subjects were male and had acceptable levels of mathematics preparation and skills. Each subject, however, was available for *only one hour.* During this hour, each subject first completed a roughly 20 minute "Math Affect" questionnaire (Anderson, 1981; Allen & Carifio, 1999a) comprised of five subscales and a specially development 35 bipolar pairs semantic differential that measured the three meta-cognitive judgments of quality, relevancy and proximity which according to Polya lead to emotions during (complex) problem solving and the emotional and self-reported physiological were feeling in real time while solving problems (see Allen & Carifio, 1999b).

Each of these five factors was supported by principal component factor analyses which account for 70% to 83% of the variance. This semantic differential was completed by subjects multiple times while working problems (see below for details).

Each subject completed 2 Polya problems, one easy and one hard problem, during the 35 minute period that was available for them to work on the problems. We most certainly would have liked to have had more time so subjects could have done 4, 6 or even 8 problems, which would have required a maximum of 160 minutes per subject to do a whole 8 problem set, but we had to conduct our exploratory inquiries in the 60 minute time period that were very lucky to obtain in this setting.

Given that many research studies published in excellent journals sometimes have the scores on one or two Likert items as the dependent variable (e.g., Thiede et al, 2003), or the truncated rubric scoring of a single free written or verbal response (e.g., Dinsmore et al, 2008) or problem (e.g., the Tower of Hanoi and The Tower of London), we believe that the analytical scoring of two Polya problem solutions, each of which took 10 to 25 minutes to produce, along with the data from 3 (or more) semantic differentials that subjects completed during each problem solving period, is more than sufficient as both criterion and process measurements and as a research design for an exploratory pilot study of the efficacy of the problems constructed and the theory devised, when put into an appropriate, reasonable, and realistic contextual perspective, and particularly so when compared to many published case studies (e.g., Hubbs & Brand, 2005). The actual amount of time allotted to the more difficult Polya problem in this exploratory study (20 minutes), it should be noted, is approximately the same amount of allocated working time used for the classic Tower of Hanoi problem in literally hundreds of studies (see Kotovsky et al, 1985; and Hinz et al, 2013).

The 2 Polya problems used in this exploratory pilot study were randomly selected from the 24 problems developed. The first problem was a traditional algebra world problem and the second problem was a novel, unconventional problem designed to challenge students and their problem solving skills. The two problems were administered to subjects in random order. The same two problems were used for all subjects rather than giving subjects random easy-hard (or the reverse) pairs from the 24 problems developed to keep this factor constant in this study so as not to introduce extraneous problem differences variance into the data as well as to keep the sample size for analyses large in this study. All data were collected from subjects in groups of 25 in a large and comfortable classroom. The second author of this article scored the problems with the California holistic scoring rubric. Prior to analyses a subject's total score on both problems were used to divide subjects in high (good), medium and low (poor) problem solvers groups.

To sample moment to moment changes in meta-cognitive evaluations, emotions, and self-reported physiological responses, subjects completed the semantic differentials 6 times while trying to solve the two problems (before, midway and after each problem) except if they were exceptionally quick in solving the problem in which case the midway measurement was not obtained (a rare occurrence). Completing the semantic differentials took about two minutes on each occasion. This design was necessary for several access, scoring burdens, and time-restriction limitations, as well as other factors. Further, due to this design several analyses that are often done with more traditional types of "test items' could not be done. However, it should be noted that the effects observed in this design are *lower order estimates* of effects that would be observed over a complete subset of 8 Polya problems or all 24 problems, and that this point is particularly true of the cumulated (emotional) effects that would be observed over doing the "hard" subset of Polya problems as opposed to just one.

7.0 Results and Conclusions

Our initial research (see Allen & Carifio, 2004 for these details) showed that these two Polya problems significantly differentiated between good and poor problem solvers ($p<.0001$) and their associated meta-cognitive and emotional levels. This initial research also showed that the hard problem was significantly more difficult to solve ($p<.001$) than the easy problem, which supported the classification criteria we used and the classifications we made. This initial research also showed subjects who got a problem correct reported more positive emotions and lower self-reported negative physiological responses than subjects who did not get the problem correct. Subjects who did not solve the problem correctly reported higher negative emotions and higher self-reported negative physiological responses. This pattern was particularly true for subjects who got both problems assigned to them correct or incorrect.

As subjects had a hard and an easy problem to solve and the 2 problems were randomized as to which problem they did first, these results along with several others (see Allen &Carifio, 2007 for details) support various aspects of Polya's theory of problem solving, as well as Mandler's theory about emotion in problem solving in terms of the effects of emotional disruption and oscillations during mathematic problem solving, and that mathematically more sophisticated and accomplished people have more intense and more highly differentiated emotions, and particularly so in terms of level of cognitive development, which is a claim and view of [mathematical] problem solving one only finds in Polya.

There were high positive correlations between the meta-cognitive evaluations subjects reported making and the emotions and physiological responses they reported having (r=+.73 to +.93).

In terms of Polya's three meta-cognitive evaluation factors, the quality factor accounted for 7 times the variance of the relevance and the proximity factors in all analyses. Further, one particular set of findings was highly suggestive and may be important in several different ways. The non-orthogonal factor analysis results of all emotion related items showed that the *self-reported* physiology items accounted for only four percent of the total variance in the self-reported emotion responses. Because emotion currently is viewed by many theorists and researchers as having both cognitive and physiological ("in the body") components that interact with each other (Mandler, 1989; Tarvis, 2011), this result implies that *perceived* physiology is a weak contributor to the intensity and quality of emotion reported by subjects for these kinds of problem solving experiences. This finding contradicts the conventional classical view that the physiological component of emotions account for and explain more of the variance in emotions than the cognitive component (e.g., Thorndike, 1911; Gagne, 1985; and Carver &Harmond-Jones, 2009), which is a view that is currently the dominant view among many mathematics educators, psychologists, and neurologists. This finding of the cognitive component of emotions being far stronger and explaining much more of the variance in emotions than the physiological component supports Polya and Mandler's view of emotions as well as many current proponents of the cognitively oriented two component view of emotions (e.g., Meyer, 1998; Tarvis, 2011; and Moors, 2013), which is exactly why our finding should be further investigated.

We would be remiss if we did not point out here that there are a number of disagreements and controversies about the point made above; namely, that the cognitive factor is far more powerful and causal in emotions that the physiological factor (see Coan, 2010; and Fischer et al., 2012), even though this view has been Tarvis' (1989, 2011) central and well supported general point (except for extreme clinical cases) for over two decades. Therefore, our finding that supports the view of Travis, Mandler, Meyer, and Polya is an important finding that would seem to merit further and closer investigation given the results of this study and particularly as it is currently a distaff view in this area.

Further, it should be additionally noted that there is also a current view (Azevedo, 2009; and Camras, 2011) that the cognitive and physiological components of emotions are actually a unity that cannot be separated (or manipulated independently) and discussions of differences between the two components are moot, as any differences between the components come about as a results of differences in initial starting conditions between them in a given context (a loose "chaos theory" or oblique factor analytical view). While conceptually this latter view may indeed be the case, there is not yet any clear and strongly convincing data to support an emergent unity, reciprocal determinism, and/or subject to initial conditions view of emotions, broadly and across domains, in terms of this "in-the-mind" and 'in-the-senses" *simultaneously* phenomenon making the three views presented here somewhat equally plausible and somewhat equally acceptable and usable as of now. Additionally, both correlated as well as independent factor models, and chaotic dynamic over time, may be easily handled by multiple regression models and other multivariate models, as well as non-linear (chaotic) multiple regression models of the theoretical kind developed by Rene Thom (1975) and Zeeman (1976). For a concrete example of these two latter points for anxiety, emotion, [mathematical] problem solving, and chaotic dynamics over time, the reader is referred to Allen and Carifio (1995) which supports the classical view of these variables used here, as the view one uses is basically an example of the axiom of choice in mathematics and research as to which view one choses to conceptualize, measure, analyze and interpret the data. Therefore, Polya's and Mandler's two factor classical view of emotions is a perfectly acceptable view for both their theories and the testing of them, and as compared to other views extant in the literature.

Implicitly, the cognitive view of emotion (particularly in a therapeutic context) holds that cognition accounts for the major portion of variance in emotion and that physiology is a far less powerful contributor for most people in most (i.e., non-extreme) contexts. Further, Polya's contention and argument *is that more sophisticated and accomplished people have more intense and more highly differentiated emotions, and particularly so in terms of level of cognitive development.* This argument follows logically as an implication of both the cognitive view and the findings of the factor analysis. This view and contention of Polya that more sophisticated and accomplished people have more intense and more highly differentiated emotions, and particularly so in terms of level of cognitive development is a view that we confirmed empirically in a separate study we published (see Allen & Carifio, 2007 for details).

Polya did not explicitly state this view or its logic, and was to some degree trying to account for the qualitative differences in emotion across levels of human development and education.

But the cognitive view is inherent in Polya's model and throughout his writings and the results summarized here and reported in detail elsewhere (see Allen &Carifio, 2004 and 2007) strongly support this view. That cognition may be much more important than physiology in emotion in contexts similar to the ones in this study is a finding that needs to be researched further because of its importance and far reaching implications particularly *instructionally*. Not even Mandler, a respected emotion theorist in mainstream academic psychology, has addressed the relative weights and importance of cognition and physiology in his two factor model of emotion nor have other alternative theorists either. Polya not only implicitly addressed this question but also provides a more differentiated model of cognition and emotion.

It is commonly believed that the main effects of emotion in problem solving are disruptive and distracting and that negative emotions interfere with and diminishes performance. The actual effects of emotion in problem solving and the evidence to support these actual effects have been summarized here. Instead of diminishing performance, the evidence and results of this study (and others we have done) clearly indicate *that positive emotions energize, organize, focus, and improve performance* as Polya contended, as has also been shown in studies of hope and optimism of at-risk learners in engineering and science courses (see Carifio & Rhodes, 2002 for details). Furthermore, negative emotions provide highly valuable problem solving information for problem solvers in general and for sophisticated problem solvers in particular. These findings have immediate implications for the wide-spread "feel good" approaches used to teach mathematics during the last decade. In particular, these teaching approaches were misdirected in their over-generalized and undifferentiated views about emotion and its role in problem solving and learning. Cognitive dissonance and emotional conflict, *within appropriate limits*, have very positive psychological and learning functions and are not inimical factors to be eliminated from the learning process. Also appropriate levels of stress both during and after learning are essentially for long term memory formation and consolidation (see Meeter and Mure, 2004, for details). Allowing students to have both positive and negative emotions as an integral part of mathematics learning is not necessarily demeaning or detrimental. Instead, according to these results, a varied-emotion approach generates valuable meta-cognitive evaluation information as well as stimulating, energizing, organizing, and focusing effects from the concomitant emotional outcomes.

The key to success in problem solving, according to the results summarized here, comes when students consider the cognitive evaluations generated during problem solving to be personally important and then when they constructively utilize the differentiated emotions that occur as a result. Provided that the importance of cognitive evaluations and concomitant emotions has been taught as part of a general model of problem solving practice, the use of authentic, challenging, and intrinsically relevant problems, where students have a personal stake in the outcome, may be particularly beneficial in helping students become better problem solvers. Lastly, the results presented here indicate that less sophisticated problem solvers might gain valuable problem solving experience from having more sophisticated problem solvers act as role models and peer mentors in cooperative learning situations. This mentoring approach might be most effective, from a long-term benefit perspective, when applied in middle school contexts where students encounter elementary but real Polya Problems for the first time (hopefully).

Informal analyses using the Polya Model Answer for each problem indicated that the stages and key functions in problem solving identified by Polya were observable in the responses of these subjects but that a more standardized and detailed response format and analytical scoring rubric would be needed to assess these “process aspects” and the non-linear emotional “disjuncture points” during problem solving better.

However, in terms of our central thesis about "Polya Problems," the associated corollaries of this thesis, and the need for a standard set of Polya Problems to study more complex mathematical problems solving, the results of these initial studies support our views, and that the 24 problems developed and the holistic rubric presented here for scoring them are such problems and have initial exploratory validity. We are currently collecting data on these and other problems to explore issues of problem difficulty, methods of scoring, reliability, validity, and test variance.

We would like other researchers and program evaluators to use our Polya Problems to both replicate, extend, and supplement our findings. In particular, we would like to see studies that have subjects do one or more of the subsets of 6 to 8 Polya problems, so more sophisticated analyses and studies may be done of the subset(s) and their equivalencies as well as of mathematical problem solving and emotion during mathematical problem solving. We would also like to see studies done that explore more sophisticated analytical scoring rubrics for these Polya problems and observe and interview students both while and after they do these Polya problems about their processes and strategies and emotions. Scoring Polya problems with Model Polya Answers, it should be noted, is a task that is very similar to scoring geometrical proofs (see Carifio & McBride, 1997 for details).

We would also like to see studies that teach Polya's problem solving approach and methods to students, and how effective this strategy would be in improving the mathematical knowledge, understandings, skills and problem solving abilities of middle school to college students. We would also like to see studies that teach and develop the on-going real time meta-cognitive skills of quality, relevancy and proximity that Polya has identified and we have validated, and the effects the development of such skills (in middle school to college level students) would have on the acquisition of mathematical knowledge, understandings, skills, and problem solving abilities. We also believe that these later meta-cognitive skills will be shown to be general problems solving skills and one with lasting value in future studies that are done on this particular point and hypothesis.

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