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Assessment of Taiwanese Students' Conceptual Development of Fractions

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Abstract

The purpose of this study was to apply exploratory latent class analysis to a conceptual test of fractions supplemented with well-defined, cognitive information in order to understand strengths and weaknesses of Taiwanese elementary students' conceptual development of fractions. Data were collected from 2612 fifth and sixth graders in Taiwan in 2002. Results indicated that there were five latent classes behind the responses. The majority of differences among the five classes were quantitative in nature, but qualitative differences were evident between two moderate performing groups. Each of the latent classes showed cumulative difficult areas in terms of required cognitive skills, some more than others. The diagnostic information in terms of strengths and weaknesses of fractional contents and skills was discussed.

Keywords: cognitive skill profiles, conceptual learning of fractions, educational assessment, exploratory latent class analysis

1. Introduction

Classifying students into different groups based on their different types of performance on cognitive test items is important for providing adaptive instruction or learning environments. Previous studies have identified differences between students' performance levels and problem solving strategies. For instance, Aksu (1997) discussed a pattern of proficient students focusing more on the structure of the problem, while students with lower performance tended to focus on more superficial information. Behr, Wachsmuth, and Post's (1985) study showed a pattern of proficient students incorporating estimation in solving fraction problems. Their responses to problems, such as fraction order and equivalent fractions, seemed to come more naturally and showed more flexibility with the process, suggesting the presence of conceptual knowledge. Cole and colleagues found that higher functioning students gained more from integrated environments, while lower functioning students benefited much more from segregated settings (Cole, Mills, Dale, & Jenkins, 1991). In other words, the crucial issues for providing adaptive instruction or learning environments are how students are classified into different groups and how characteristics of these groups of students are distinguished appropriately. Generally, students are often classified into different groups based on total performance on achievement tests (Chan, Leu, & Chen, 2007) or overall performance in a particular discipline (e.g., mathematics and/or science, or reading comprehension). This type of classification may raise instructional obstacles because even though students in the same group may have similar achievement levels, they do not necessarily have the same strengths and weaknesses on various disciplines or in a particular discipline.

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Inappropriate grouping may lead to the failure of providing adaptive instruction. Speece (1990) reviewed research literature regarding interactions between student aptitude and instructional treatment and concluded that much empirical research of aptitude-treatment interactions (ATIs) failed because of groups of students with heterogeneous aptitudes. Many diverse statistical approaches such as factor analysis, discriminant function analysis, cluster analysis, and latent class analysis have been used to classify students. Because latent class analysis (LCA) has several statistical advantages, such as availability for alternative model comparisons (Vermunt & Magidson, 2002a), much research applies latent class analysis to yield distinct subgroups. For example, Brown (2007) compared cut-score results between latent class analysis and judgmental methods (e.g., Angoff cut-score method) and results supported the use of LCA in grouping performance ability groups. Vermunt and Magidson (2002b) conducted a simulation study to compare the LCA method with cluster analysis. They found the LCA method outperformed cluster analysis even though the simulated data favored cluster analysis. This classifying membership approach is also called *exploratory* latent class analysis (ELCA) because examinees are classified into similar item response classes based on their item response patterns first and then group member characteristics are extracted from item properties of each class. In other words, the number of latent classes is determined first, and then external factors may be found to link with these classes for the possible interpretation of heterogeneous subpopulations (Baghaei & Carstensen, 2013). However, extracting useful or diagnostic information from each class is not an easy task for the researcher because of limited access to test items (e.g., Baghaei & Carstensen, 2013) or the nature of disciplines. As in the Baghaei and Carstensen (2013) study, failure to have access to the actual reading items is considered a limitation because with only the test specification available, they extracted passage length as an external factor associated with two latent classes found in the study. The information regarding passage length with latent classes may not be useful for many researchers or practitioners. They highly recommended detailed examinations of test item contents that may extract "deeper insight into development and processing involved in reading comprehension" (p. 9). Thus, this study applied exploratory latent class analysis (ELCA) to a conceptual test of fractions supplemented with a well-defined cognitive model in order to provide group-wise diagnostic information in terms of Taiwanese students' performance on fraction learning. To present the technical background for this study, three areas are reviewed. First, latent class analysis and its underlying equation are elaborated. Second, cognitive models that provide more detailed information on the tests are briefly introduced. Third, the linkage of latent class analysis with detailed cognitive information is newly developed.

1.1 Latent Class Analysis

Latent class analysis was first introduced by Lazarsfeld and Henry (1968) to explore latent, categorical attitude variables. The LCA method is a statistical modeling technique that categorizes respondents into distinct classes or subgroups (Hong & Min, 2007). These distinct classes or subgroups are called latent classes because they are distinguished based on unobservable sources (Yang, Shaftel, Glasnapp, & Poggio, 2005). Unlike factor analysis that assumes factors as continuous latent traits, LCA considered subgroups as categorical latent classes.The underlying principle behind LCA is that the respondents are assumed to be drawn from a heterogeneous rather than homogeneous population. Hence, the aim of latent class analysis is to classify respondents intodistinct latent classes based on their responses to the items and to identify items to characterize latent classes (Nylund, Asparouhov, & Muthén, 2007). A number of the LCA mathematical models with various statistical algorithms and computer software have allowed for applications in many substantive fields with different types of outcomes (e.g., binary, ordinal, and continuous) or any combination of them (Nylund, Asparouhov, & Muthén, 2007). Since the data used in the present study were dichotomous, the mathematical expression of the LCA model with dichotomous data is described below. There are two types of parameters that need to be estimated in the LAC models: item probability parameters and class probability parameters. The item probability parameter, denoted P_{ik}, in the LCA model with dichotomous data is the probability of a person answering item i (i = 1, 2,…,n) correctly given that the person is in the k^{th} latent class (k = 1, 2,…, K).The class probability parameter, denoted P_k, is the probability of being the k^{th} latent class. The class probability parameter can be referred to as the relative prevalence (or the size) of each class (Nylund, Asparouhov, & Muthén, 2007) We denote u_i = 1when item i is answered correctly and u_i = 0 otherwise. Hence, the probability of answering item i correctly (P_i) is

$$P_i = P(u_i = 1) = \sum_{k=1}^{K} P_k P(u_i = 1 \mid k) = \sum_{k=1}^{K} P_k P_{ik} \quad (1)$$

Assuming conditional independence, the joint probability of a given item response pattern (*U*), denoted *P*(*U*), is

$$P\,(U) = P(u_1, u_2, ..., u_n) = \sum_{k=1}^{K} P_k \prod_{i=1}^{n} P_{ik}^{u_i} (1 - P_{ik})^{1-u_i} \qquad (2)$$

1.2 A Cognitive Model of Fractions

The cognitive model utilized in educational assessment is to describe students' problem-solving and thinking processes on academic tasks. In other words, a cognitive model "helps to characterize the knowledge and skills students at different levels of learning have acquired and to facilitate the explanation and prediction of students' performance" (Leighton & Gierl, 2007, p. 6). Hence, the cognitive model can make test scores more interpretable and meaningful (Cui & Leighton, 2009). One of the popular cognitive models based on Leighton and Gierl's (2007) taxonomy is generated on the basis of curriculum and may include a comprehensive set of representative concepts and skills that educators or practitioners think students should master in a particular domain of interest (for more discussion of other cognitive models, the reader is referred to Leighton and Gierl, 2007). An illustrative example Leighton and Gierl (2007) provided is a list of sequential concepts and skills considered for an elementary-level curriculum-based mathematics test adapted from Idol's (2007) models of curriculum-based assessments. For instance, for the domain of Sets and Numbers a list of the sequential knowledge and skills may include sets, counting, comparison, odd-even, ordinal numbers, bar graphs, and so on. In the few LCA studies regarding mathematics achievement, Yang and colleagues (2005) used the existing Kansas curriculum standards such as content areas (i.e., number and computation, Algebra, Geometry, and data)and benchmarks(e.g., probability and statistics for the data content) to describe the characteristics of mathematics test items used in their study.

But the cognitive information on test items has never been applied to the interpretation of the LCA findings. In contrast, Chan, Leu, and Chen (2007) identified five common latent factors based on exploratory factor analysis (EFA). Although they do not mention anything regarding cognitive information or models in the study, these five latent factors represent five fraction abilities and are used to help interpret the LCA findings. These five abilities involve an ability representing the Equal Sharing skill, two abilities representing the Unit skills, and two abilities representing the Equivalent Fractions skills. The two Unit abilities are one general unit ability and one specific-unit ability that relates to the conditions when the whole consists of multiple units. The two Equivalent Fractions abilities are one general equivalent fractions ability and one specific equivalent fractions ability that conducts "a fractional comparison with at least one denominator not being a multiple of twos" (Chan, Leu, & Chen, 2007).However, these latent factors or abilities based on exploratory factor analysis may not completely extract all cognitive knowledge and skills that students use to solve test items. The cognitive model of fractions used in this study included four content areas (i.e., simple fraction, equal sharing, unit, and equivalent fraction) with a list of cognitive skills. Even though each item on the test can be classified into one or more contents and skills, only one major content area and cognitive skill were extracted from each item. Task analyses conducted by mathematics educators and in-service mathematics teachers in Taiwan were used to validate content areas and cognitive skills for test items. Several students from the sample were asked to think aloud regarding their solutions so as to provide further validity evidence of primary cognitive skills. The content areas and cognitive skills generated as the cognitive model of factions in this study are presented and described in detail in the method section.

1.3 Linking LCA with the Cognitive Model

The way to link distinct latent classes with the cognitive information is to apply acutoff value to item location (difficulty) parameters or the proportion of correct responses in latent classes to determine if the examinees in a latent class master the corresponding cognitive skill of an item. Since the latent class analysis in the WINMIRA software uses the Raschmodels to define the probability of a correct response (Von Davier, 2001), the proportion of a correct response for a particular item is a sufficient statistic for estimating item difficulty for that item. For instance, the correct proportion of .50 that indicates a half of examinees in a latent class answering an item correctly is equivalent to the item difficulty parameter of 0.Furthermore,only one dominant cognitive skill is extracted for an item in the cognitive model for a test. Thus, if the correct proportion for a particular item in a latent class is greater than the cutoff value, it is assumed that the examinees in that latent class master the corresponding cognitive skill for that item.

1.4 Research Purpose and Questions

It should be noted that the data used in this study had been analyzed by Chan, Leu, and Chen (2007).

They proposed a two-component multidimensional item response theory (MIRT) model with latent class analysis (using MPLUS software) to explore students' conceptual deficiencies of fractions in a group-wise fashion. In this study, we intended to apply latent class analysis using a commercial program called WINMIRA (von Davier, 2001) and supplement with a well-defined cognitive model of fractions to better understand strengths and weaknesses of Taiwanese elementary students' conceptual development of fractions. Latent class analysis was used to classify students into distinct latent classes based on their item responses. Students in the same classes have similar characteristics of conceptual understanding of fractions whereas those in the different classes have different characteristics. Thus, the first and most important question was how many distinctive latent classes were found within the sample of Taiwanese students. After the distinctive latent classes were identified, the profiles of the class-specific item difficulties across twenty-three items were examined to indicate whether the differences among these distinctive latent classes are qualitative or quantitative in nature. That is, the second research question was if the profiles of item difficulties across test items for distinct latent classes had parallel (quantitative) or crossing (qualitative) patterns. Third, in conjunction with a well-defined cognitive model, we aimed to explore the strengths and weaknesses of conceptual learning of fractions for students in each latent class as diagnostic information. The third research question was what strengths and weaknesses of fractions Taiwanese students had. Finally, mathematics curricula and instructional activities in Taiwan were discussed with these findings in this study.

2. Method

1.1 Participants

Efforts were taken to maintain a sample representative of the population in Taiwan. A total of 42 schools and 84 classrooms were sampled and 2,612 students provided data in 2002. There were 1,283 (49.12 %) fifth graders and 1,329 (50.88 %) sixth graders. The gender distribution consisted of 1,330 (50.92 %) boys and 1,282 (49.08 %) girls.

2.2 Instrument

Twenty-three multiple-choice items of the conceptual test of fractions developed by Chan and Leu (2004) were used in this study (available on request). The fractional items were designed to measure four content areas: basic fraction (1 item), equal sharing (3 items), unit (7 items), and equivalent fraction (12 items). The Cronbach alpha for the entire test was .85.

2.3 Item Content and Cognitive Skills

Since student construction of fractions is the by-product of mathematics curricula and instructional activities (Charalambous & Pitta-Pantazi, 2007), the conceptual test of fractions used in this study was designed mainly based on mathematics curricula, teaching materials, and mathematics educators' research findings in Taiwan. The basic fraction item was to examine the understanding of basic meanings of numerator and denominator in the fractional notation. The equal sharing items were to examine the understanding of the equal parts into which the whole is partitioned. The unit items were to examine the understanding of the unit notions in a variety of conditions (e.g., forming a fraction with two different units or two one-units, comparing fractions with unknown or different units, etc.). The equivalent fraction items were to examine the understanding of equivalent fractions between numerical fractions and graphic representations and between two different units. Table 1 lists specific cognitive skills identified by mathematics educators and in-service teachers for each content area and corresponding items.

Table 1: Four Content Areas, Cognitive Skills, and Corresponding Items

Content	Cognitive Skill	Item
Basic Fraction	BF: Be able to form the fraction when the parts and the whole are given explicitly	4
Equal Sharing	ES: Be able to discern if the whole has been partitioned into equal parts or be able to partition the whole into equal parts	1, 2 10
Unit	U1: Be able to form the fraction when the whole equals the unit	16
	U2: Be able to for the fraction when the whole equals two one-units	13, 15
	U3: Be able to compare fractions with unknown units	
	U4: Be able to compare fractions with different units	7, 23 12, 14
Equivalent Fraction	EF1: Be able to identify the visual representation of equivalent fractions when the number of equal parts of the graphic representation are two times of the denominator of a numerical fraction	3, 6, 9, 11
	EF2: Be able to identify the visual representation of equivalent fractions when the denominator of a numerical fraction is two times of the number of equal parts of the graphic representation	5, 8
	EF3: Be able to compare fractions when one denominator is two times of the other.	
	EF4: Be able to compare fractions when one denominator is not being two times of the other.	18, 19 17, 20,21, 22

2.4 Statistical Analysis

To address the research purpose and to answer the questions in this study, the following analytical steps were conducted. First, several models with different numbers of classes were tested to determine the appropriate number of latent classes that fit the data using WINMIRA (von Davier, 2001), which is commercial software that can be purchased from software distributers (e.g., Assessment Systems Corporation in U.S.). Since competing models in latent class analyses were not nested, indices used to determine the model fit were the Akaike information criterion (AIC), the Bayesian information criterion (BIC), and the Consistent Akaike information criterion (CAIC). All three were used to address similarly issues of goodness-of-fit and parsimony with the smaller values indicating a better fitting class model (Yang, et al., 2005). Second, once the number of latent classes was determined, the nature of each class was explored by checking the class-specific item difficulty profiles across the 23 items. The class-specific item difficulty parameters were computed for each latent class. Therefore, the item difficulty profiles were compared among classes to find if the profile differences across classes were of a qualitative or quantitative nature (Embretson & Reise, 2000). When the item difficulty profiles for two classes are parallel and item difficulties for one class are consistently larger than those for the other class, a quantitative ability difference exists between these two classes. It indicates that one class has higher ability levels on all the testing domains and cognitive skills than the other. However, if crossing occurs, the differences are of a more qualitative nature. It indicates that these two classes may demonstrate different combinations of ability levels inthe content areas and cognitive skills. In other words, given the content areas of fractions used in this study as an example, one class may show high knowledge levels on the equal sharing concept but the other may exhibit high knowledge levels on the equivalent fraction concept. Third, class-specific item difficulties and category probabilities along with content areas and cognitive skills were used to characterize each latent class. Category probability provided in WINMIRA for binary data represents the proportion of students in a latent class answering an item correctly. In this study, the category probability of .70 was set as a cutoff value for each item to determine if a cognitive skill underlying an item was mastered by examinees in a latent class. In other words, if an item has a correct probability equal to or greater than .70 in a latent class, it is assumed that students in that class master the skill underlying that item. Based on these mastery/nonmastery skill profiles, the strengths and weaknesses of latent classes were identified.

3. Results

3.1 Number of Latent Classes

The number of the latent classes for student performance on the conceptual test of fractions was determined by first checking fit indices, including AIC, BIC, and CAIC. Latent class models with one to six classes were used to fit the fraction conceptual data. Table 1 reports the fit indices information for six latent class models. Results showed that both the BIC and CAIC values for the five-class model were smallest but the AIC value indicated the six-class model. Baghaei and Carstensen indicated that "*AIC* is not asymptotically consistent as sample size is not used in its calculation. BIC and CAIC penalize more for the number of parameters and therefore chooses the models with fewer parameters compared to AIC" (Baghaei & Carstensen, 2013, p. 5). Further, more consistency and parsimony were found with BIC and CAIC in previous research (e.g., Li &Nyholt, 2001; Nylund, Asparouhov, &Muthén, 2007), so the BIC and CAIC fit information criteria were used over the AIC in the study. The five-class model was chosen as the best-fitting solution for our data.

Table 2: Model Fit Indices for Six different Latent Class Models among All Participants

Model	AIC	BIC	CAIC
1-class model	69153.48	69288.44	69311.44
2-class model	60715.36	60991.15	61038.15
3-class model	59349.73	59766.35	59837.35
4-class model	59060.77	59618.21	59713.21
5-class model	58842.32	**59540.60**	**59659.60**
6-class model	**58769.99**	59609.09	59752.09

Note: Values in bold indicate the best-fitting solution.

We further examined classification quality for the five- and six-class solutions. The computer program WINMIRA (von Davier, 2001) provides the output, named statistics of expected class membership that include mean classification probabilities of diagonal and off-diagonal elements in the matrix, for examining classification quality. As mean probabilities of diagonal elements are higher and, in contrast, mean probabilities of off-diagonal elements are lower, classification quality is better (Hong & Min, 2007). Table 3for the five-class model shows that all diagonal mean probabilities were higher than .80 and all off-diagonal mean probabilities were lower than .16. However, only two out of six diagonal mean probabilities for the six-class solution were higher than .80 (the table available on request). Therefore, the indicator of classification quality favored the five-class solution for our data as well.

Table 3: Expected Class Membership for the Five-Class Model

	Mean classification probability				
Class	1	2	3	4	5
LC1	**0.844**	0.155	0.001	0.000	0.000
LC2	0.064	**0.833**	0.098	0.004	0.000
LC3	0.000	0.125	**0.817**	0.049	0.009
LC4	0.000	0.011	0.076	**0.805**	0.108
LC5	0.000	0.000	0.015	0.084	**0.902**

Note: Values in bold indicate mean classification probabilities of diagonal elements.

3.2 Nature of Latent Classes

Given that the five-class model was chosen, Figure 1 shows the profiles of the difficulty parameters across 23 fraction conceptual items for the five latent classes. Items with low (or negative) values (i.e. -1, -2, -3) of class-specific difficulty parameters indicate relative ease for examinees in choosing the correct responses, as compared to high (or positive) values (i.e. 1, 2, 3) indicating more difficulty in selecting the correct responses. Since the profile lines across 23 items in Figure 1 seemed to be parallel, the differences among five distinct latent classes appeared to be primarily quantitative, suggesting that students in one class consistently performed higher or lower than those in other classes. The line at the bottom with the lowest difficulty values represented the best performance latent class where the line at the top with highest difficulty values was the poorest performance latent class.

Therefore, the line at the bottom was arbitrarily denoted Latent Class 1 or LC1, representing the highest performance group. The second line from the bottom, denoted Latent Class 2 or LC2, was the second highest performance group.

The line at the top, denoted Latent Class 5 or LC5, was the lowest performance group. The lines between LC2 and LC5, denoted Latent Class 3 (LC3) and Latent Class 4 (LC4), were the moderate performance groups.

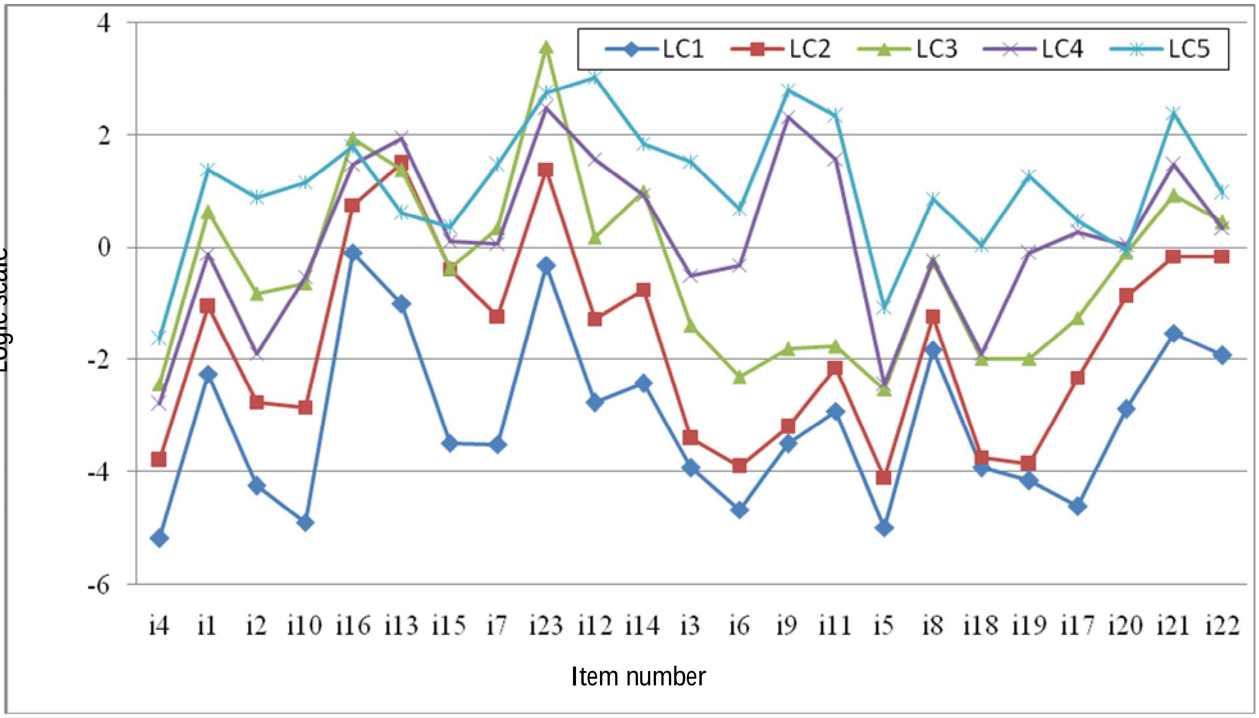


Figure 1: Item Difficulty Profiles for the Five Latent Classes

The WINMIRA software also provides descriptive statistics for each latent class (e.g., raw score range, mean score, category probability for each item) that can be used to support five latent categories based on item difficulty profiles mentioned earlier. Table 4 presents these statistics with class size for each class. For instance, most students in LC1 tended to have the highest raw scores ranging from 19 to 23 out of 23 points and the average score was 20.85. Twenty-one out of 23 items had over .70 proportions of a correct response in this class. There were around 10% of students in this study classified into the highest fraction knowledge group.In contrast, students in LC5, the lowest performance group, had an average score of 6.55, ranging from 4 to 9. There were only 2 items that had correct proportions over .70 and 17.3% of students were classified in this group.

Table 4: Class Size, Score Range, Mean, and Number of Items for Each Class

Class	Class Size	Score range	Mean score	Number of items*
LC1	10.0%	19 to 23	20.85	21
LC2	35.1 %	15 to 19	17.34	15
LC3	24.8%	11 to 16	13.33	8
LC4	12.8%	8 to 13	10.61	4
LC5	17.3%	4 to 9	6.55	2

Note: Number of items represents the number of items in each class that had a correct category probability greater than .70.

In addition, it is worth noting that there were some crossings between LC3 and LC4. This indicated that the differences between these two latent classes were qualitative in nature, meaning that one latent class performed better on some particular items and the others did better on other items.

3.3 Characteristics of Latent Classes

For the purpose of meaningfully characterizing latent classes, a correct category probability of .70 for each item was arbitrarily set as a cutoff value to determine if a primary cognitive skill underlying a fraction item was mastered by students in a particular latent class. A correct category probability of .70 for an item represents that 70% of students in a latent class answer that item correctly (von Davier, 2001), meaning that the skill underlying an item requires the majority of students in a latent class to correctly answer that item in order to be considered mastery for students in the class. Table 5 presents the correct category probability of each item for each latent class.

Table 5: Class-Specific Correct Category Probability of Items for Each Content Area and Skill

Content and skill	Item	LC1	LC2	LC3	LC4	LC5
BF	Item 4	0.99	0.98	0.92	0.94	0.84
ES	Item 1	0.90	0.74	**0.35**	**0.53**	**0.2**
ES	Item 2	0.99	0.94	**0.7**	0.87	**0.29**
ES	Item 10	0.99	0.94	**0.65**	**0.63**	**0.24**
U1	Item 16	**0.52**	**0.32**	**0.13**	**0.18**	**0.14**
U2	Item 13	0.73	**0.18**	**0.20**	**0.13**	**0.35**
U2	Item 15	0.97	**0.59**	**0.59**	**0.47**	**0.41**
U3	Item 7	0.97	0.77	**0.41**	**0.48**	**0.19**
U3	Item 23	**0.58**	**0.20**	**0.03**	**0.08**	**0.06**
U4	Item 12	0.94	0.78	**0.45**	**0.17**	**0.05**
U4	Item 14	0.92	**0.68**	**0.27**	**0.28**	**0.14**
EF1	Item 3	0.98	0.97	0.80	**0.62**	**0.18**
EF1	Item 6	0.99	0.98	0.91	**0.58**	**0.34**
EF1	Item 9	0.97	0.96	0.86	**0.09**	**0.06**
EF1	Item 11	0.95	0.89	0.85	**0.17**	**0.09**
EF2	Item 5	0.99	0.98	0.93	0.92	0.74
EF2	Item 8	0.86	0.77	**0.56**	**0.56**	**0.30**
EF3	Item 18	0.98	0.98	0.88	0.87	**0.49**
EF3	Item 19	0.98	0.98	0.88	**0.52**	**0.22**
EF4	Item 17	0.99	0.91	0.78	**0.43**	**0.38**
EF4	Item 20	0.95	**0.7**	**0.52**	**0.49**	**0.51**
EF4	Item 21	0.82	**0.54**	**0.28**	**0.19**	**0.08**
EF4	Item 22	0.87	**0.54**	**0.39**	**0.41**	**0.27**

Note 1: BF=Basic fraction; ES=Equal sharing; U1-U4=Unit skills 1-4; EF1-EF4=Equivalent fraction skills 1-4.
Note 2: Values in bold in each latent class indicate a correct category probability less than .70 for test items.

Based on our mastery/non-mastery cutoff value, the highest performance group (LC1), mastered all the required skills except for the skills underlying Items 16 and 23. Item 16 required the skill of being able to form the correct quantities of the whole and the part when the whole equals a unit, which is a basic skill of the unit content. However, Item 16 also required a careful reading to notice two objects (marbles and buttons) in the prompt. Buttons in this item were regarded as "*redundant information*".Item 23 required the unit skill of being able to recognize two "*unknown*" units when comparing to each other. Hart (1981) indicated that the items that require the student to apply uncertain conditions are the most difficult fraction questions. More importantly, item 23 also heavily depended on writing (communicating) ability, compared to Item 7 that required the same unit skill but provided explanations in options. Walker and Beretvas (2003) argued that mathematics items requiring the communication skill may result in the second dimension and become more difficult items. This indicated that Taiwanese students lacked a writing communication skill in mathematics education even with students in the highest performance group. Overall, students in LC1 seemed to master all the content areas and skills in terms of fractions.

What weaknesses students in LC1 had was the writing communication skill and carefulness. Further, these two weaknesses were possessed by students across the five latent classes.

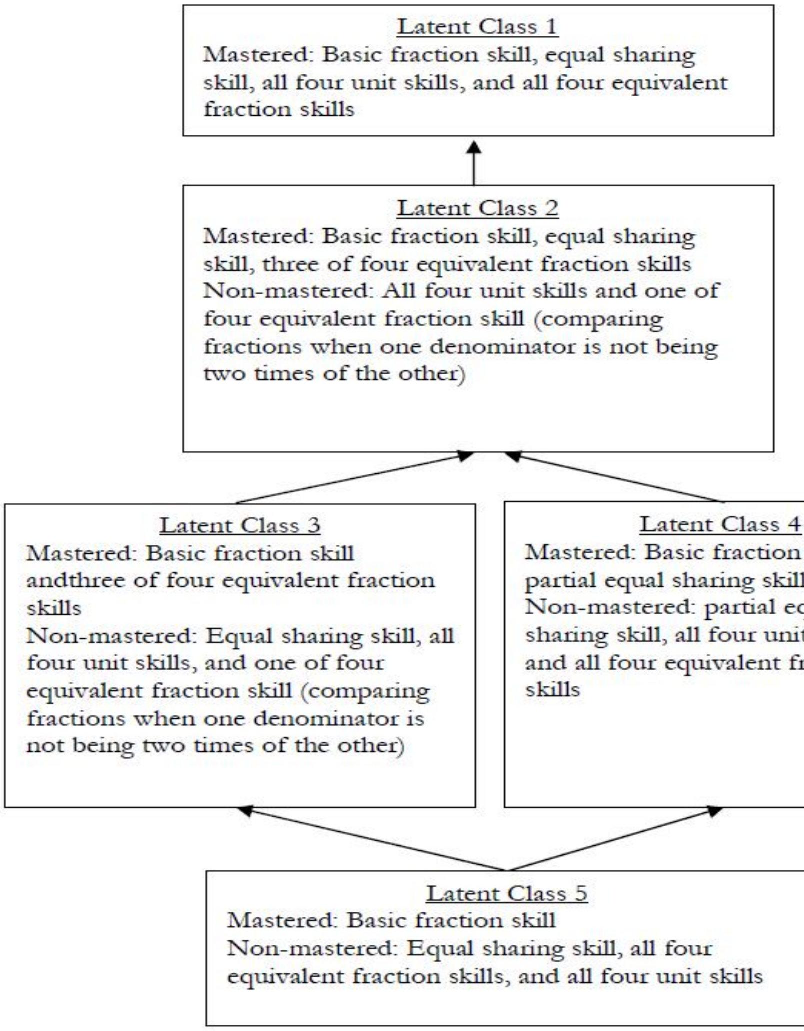
The second highest performance group (LC2) mastered all the primary skills except for the skills underlying Items 13, 14, 15, 16, 20, 21, 22, and 23 that had the correct category probabilities equal to or less than .70.Among these non-mastered items, Items 13, 14, 15, 16, and 23 involved all four unit skills and Items 20, 21, and 22 required one of the four equivalent fraction skills, comparing fractions when one denominator is not being two times of the other. This most difficulty skill of equivalent fraction corresponded to a specific equivalent fraction ability defined by Chan and colleagues (2007). Generally speaking, students in LC2 did not master all the unit skills as well as the most difficult equivalent fraction skill. Students in one of the moderate performance groups (LC3) had all the equal sharing (Items 1, 2, and 10) and unit (Items 7, 12, 13, 14, 15, 16, and 23) items with the correct category probabilities equal to or less than .70. In other words, students in LC3 did not master all the skills in the equal sharing and unit content areas. In addition, like students in LC2, students in LC3 did not possess the most difficult equivalent fraction skill, either. But students in LC3 mastered the basic fraction skill and three of the four equivalent fraction skills that corresponded to a general equivalent fraction ability in Chan and colleagues' (2007) study.

As for students in the other moderate performance group (LC4), they only had four items (Items 2, 4, 5, and 18) with the correct category probabilities greater than .70. Among these mastered items, Item 4 required the basic fraction skill, Item 2 was one of three equal sharing items, and Items 5 and 18 were two of twelve equivalent fraction items. Consistently, students in LC4 did not master any unit skills. On the basis of these mastery items, it was reasonable to conclude that students in LC4 only mastered the basic fraction skill and partially mastered the equal sharing skill. It was also reasonable to conclude that students in LC4 did not master any skills in equivalent fraction in addition to the non-mastery unit skills. As mentioned earlier, it seemed to be a qualitative difference between students in LC3 and LC4. Characteristics of LC3 and LC4 indicated that students in LC3 had better performance on equivalent fraction but those in LC4 performed better on equal sharing. Both groups consistently mastered the basic fraction skill but did not master all the unit skills. Students in LC5 had the lowest performance because there were only two items (Items 4 and 5) that had the correct category probabilities greater than .70. Item 4 was the only item representing the basic fraction skill of forming a simple fraction. Item 5 was one of twelve equivalent fraction items that required identifying the visual representation. In other words, the only mastery skill that students in LC5 possessed was the basic fraction skill.

4. Discussion

To provide the diagnostic information for adaptive remedial instruction of fractions learning, this study shows how the cognitive information that involves fractional content and cognitive skills to a comprehensive conceptual test of fractions with exploratory latent class analysis can help researchers better interpret and understand the construct nature of distinct latent classes. The fit indices of LCA indicated that the five-class model was adequate to characterize the latent structure of our data. Mean classification probabilities further supported the classification quality of the five-class model. The result, as far as finding five latent classes, was consistent with the previous study by Chan, Leu, and Chen (2007). The software WINMIRA (Von Davier, 2001) used in this study for latent class analysis obtained very similar results to the software MPLUS (Muthén, & Muthén, 1998-2004). The similarities of the LCA results from the WINMIRA and MPLUS programs were also reflected by group mean scores and proportions of group members. On the basis of class-specific item difficulty parameters, the nature of the five latent classes predominantly showed quantitative differences. The quantitative differences were reflected by the strengths and weaknesses of students' fraction learning for each class, shown by their hierarchical relationships (see Figure 2). For instance, Figure 2 illustrates an entire hierarchically ordered relationship among the five latent classes, LC 4 had a hierarchically ordered relationship with LC5 because the mastery content and skill of LC5 (i.e., the basic fraction skill) was included in the mastery contents and skills of LC4 (i.e., the basic fraction skill and the partial equal sharing skill).

Figure 2: A hierarchically Ordered Relationship among the Five Latent Classes



The hierarchically ordered relationship also reflects the learning paths in terms of fractional content and skills. Figure 2 suggests that there were two learning paths in Taiwanese students. Learning Path 1 (LP1) showed that students mastered basic fraction first, followed by equivalent fraction, equal sharing, and unit, successively (i.e., $LC5 \rightarrow LC3 \rightarrow LC2 \rightarrow LC1$). Learning Path 2 (LP2) indicated that basic fraction was learned first, then followed by equal sharing, equivalent fraction, and unit, successively (i.e., $LC5 \rightarrow LC4 \rightarrow LC2 \rightarrow LC1$). These two learning paths actually result from an interesting finding in this study that there was a somewhat qualitatively different performance between students in LC3 and LC4. Basically, mathematics curriculum design in Taiwan starts teaching basic fraction content and skill first in the second grade and continues in third and fourth grades. The equal sharing content and skill are taught in second and third grades.

The equivalent fraction content and skills are mainly taught in the fifth and sixth grades. Surprisingly, even though the unit content and skills are always applied in fractional learning and teaching across various grades, the unit topic has not been included in Taiwanese curriculum standards (Chan, Leu, &Chen, 2007). This may explain why the unit content and relevant four cognitive skills are hardest across the five latent classes and only the highest knowledge group mastered them. LP2 matches the mathematics curriculum and LP1 occurs probably because students are more familiar with contents and skills that are most recently taught. Strengths and weaknesses of fractional contents and cognitive skills among the five latent classes provide useful and meaningful diagnostic information for remedial or supplemental instruction.

Results in this study suggest that students in LC5 who only mastered the basic fraction skill need a lot of effort to improve their learning of fractions. Based on the instructional sequence of fractions in Taiwan, teachers may focus on the equal sharing skill first, followed by the equivalent fraction skills and the unit skills. Since students in LC4 performed slightly better than those in LC5; that is, they mastered partially the equal sharing skill additionally, the remedially pedagogical procedure is similar to that for student in LC5.

As for students in LC3 and LC2, teachers may need to provide remedial instruction first on the most difficult equivalent fraction skill, which is to compare fractions when one denominator is not being two times of the other, and then followed by all four unit skills. In addition, students in LC3 need more practice on the equal sharing items because this group of students may forget the concepts and skills that are taught before. In addition to specific content areas and cognitive skills for each class, more attention needs to be paid to background knowledge, such as written communication skills of mathematical ideas or reading carefulness. Instructional strategies may include creating more learning opportunities for students to communicate their answers in writing or applying redundant units in classroom instruction when forming a fraction is taught.

5. Conclusion and Suggestions

The models of exploratory latent class analysis have been applied to research areas that mostly focus on psychological measurement such as depression (e.g., Hong & Min, 2007) or antisocial personality (e.g., Bucholz, Hesselbrock, Heath, Kramer, & Schuckit, 2000),but less often on cognitive ability tests. Apossible reason is that psychological measurement such as personality or attitude has the explicit constructs that can be readily linked to substantive interpretations of class-specific measurement items (Embretson & Reise, 2000). For instance, it has been well-known that the factor structure underlying test anxiety inventory (TAI) consists of the worry and emotionality components (Liebert & Morris, 1967). In contrast, the constructs underlying cognitive ability tests are not readily identified or lack substantive meaning (e.g., Baghaei & Carstensen, 2013; Yang, et. al., 2005). The cognitive model used in this study helps distinguish the characteristics of latent classes. The findings in terms of strengths and weaknesses of distinct latent classes utilizing a two-stage exploratory latent class analysis with a well-defined cognitive model are comparable to those from Chan and colleagues' (2007) study that applied two-components MIRT with LCA. This study even yields greater understanding of students' varied strengths and weaknesses in regard to cognitive skills with fractions. Educators and policy makers can utilize this information to provide more targeted instruction and funding, as well as programming to address the specific educational needs for the particular groups of students. In addition, there are some specific suggestions based on this study for future research. Since this study was sampled from fifth and sixth graders under the Taiwanese educational system, caution should be taken in generalizing these results to other grade levels and educational systems. For international researchers who are interested in exploring students' learning of fractions in their countries, the most important thing is to develop the cognitive model that fits curriculum standards for particular grade levels. With diverse curriculum designs, different cognitive models should be developed so that better interpretations of distinct latent classes in terms of cognitive characteristics can be made for the purpose of remedial instruction.

The research method used in this study is called *exploratory* latent class analysis because students are classified into similar item response classes first, then group member characteristics are extracted from item properties for each class. Another line for future research can be referred to as *confirmatory* latent class analysis that needs cognitive models prior to psychometric analyses. The advantages of confirmatory latent class analysis include that the more complex cognitive model that elicits all the required skills for test items can be developed and that cognitive information can be incorporated directly into psychometric analyses to provide an estimated mastery probability of each skill for an individual respondent. This type of research is warranted. Another suggestion related to both exploratory and confirmatory latent class analyses is that from a perspective of diagnostic assessments how to determine an appropriate cutoff value for skills-mastery warrants a further investigation. The cutoff values for mastery can be predominantly determined according to the defensibility and utility of the resulting interpretations like the present study and others (see Rupp, Templin, & Henson, 2010). For instance, Tatsuoka and colleagues (2004) used a single cutoff value of .70 to determine skills mastery/nonmastery. Hartz (2002) and Jang (2005) used probability bands to represent the degree of mastery; that is, they defined *mastery* as probabilities between .60 to 1, an *indifference region* as probabilities between .40 to .60, and *nonmastery* as probabilities between 0 and .4. However, these cutoff values can be further validated by conducting simulation studies that examine the reliability of correct mastery classifications (Rupp, Templin, & Henson, 2010).

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References

Aksu, M. (1997).Student performance in dealing with fractions.Journal of Educational Research, 90, 357-380.

Baghaei, P. &Carstensen, C. H. (2013).Fitting the mixed Rasch model to a reading comprehension test: Identifying reader types.Practical Assessment, Research & Evaluation, 18(5). Available online: http://pareonline.net/getvn.asp?v=18&n=5.

Behr, M. J., Wachsmuth, I., & Post, T. R. (1985). Construct a sum: A measure of children's understanding of fraction size. Journal for Research in Mathematics Education, 16, 120-131.

Brown, R. S. (2007). Using latent class analysis to set academic performance standards.Educational Assessment, 12, 283-301.

Bucholz, K., Hesselbrock, V., Heath, A., Kramer, J., &Schuckit, M. (2000).A latent class analysis of antisocial personality disorder symptom data from a multi-centre family study of alcoholism.Addiction, 95, 553–567.

Chan, W. H., &Leu, Y. C. (2004).The design of the rating scale of fraction for 5th and 6th graders.Chinese Journal of Science Education, 12, 241-263.

Chan, W., Leu, Y, & Chen, C. (2007). Exploring group-wise conceptual deficiencies of fractions for fifth and sixth graders in Taiwan. Journal of Experimental Education, 76, 26-57.

Charalambous, C. & Pitta-Pantazi, D. (2007).Drawing on a theoretical model to study students' understandings of fractions.Educational Studies in Mathematics, 64, 293-316.

Chen, Y.-H., Gorin, J. S., Thompson, M. S. &Tatsuoka, K. (2008). Cross-cultural validity of the TIMSS-1999 mathematics test: Verification of a cognitive model. The International Journal of Testing, 8, 251-271.

Cole, K. N., Mills, P. E., Dale, P. S., & Jenkins, J. R. (1991). Effects of preschool integration for children with disabilities. Exceptional Children, 58, 36-45.

Cui, Y., & Leighton, J. P. (2009). The hierarchy consistency index: Evaluating person fit for cognitive diagnostic assessment. Journal of Educational Measurement, 46, 429-449.

Embretson, S. E., &Reise, S. P. (2000).Item response theory for psychologists.Mahwah, NJ: Erlbaum.

Gorin, J. S. (2006). Test design with cognition in mind. Educational Measurement: Issues and Practice, 25, 21–35.

Hong, S., & Min, S.Y. (2007). Mixed Rasch Modeling of the Self-Rating Depression Scale: Incorporating Latent Class and Rasch Rating Scale Models.Educational and Psychological Measurement, 67, 280-299.

Lazarsfeld, P., & Henry, N. (1968).Latent structure analysis. New York: Houghton-Mifflin.

Leighton, J. P., &Gierl, M. J. (2007). Defining and evaluating models of cognition used in educational measurement to make inferences about examinees' thinking processes. Educational Measurement: Issues and Practice, 26, 3–16

Li, W., &Nyholt, D. R. (2001).Marker selection by Akaike information criterion and Bayesian information criterion.Genetic Epidemiology, 21, 272-277.

Liebert, R. M., & Morris, L. W. (1967). Cognitive and emotional components of test anxiety: A distinction and some initial data. Psychological Reports, 20, 975-978.

Muthén, L. K., &Muthén, B. O. (1998-2004). Mplus user's guide: Third edition. Los Angeles, CA: Muthén&Muthén.

Nylund, K. L., Asparouhov, T., Muthèn, B. (2007).Deciding on the number of classes in latent class analysis and growth mixture modeling: a Monte Carlo simulation study.Structural Equation Modeling, 14, 535-569.

Rost, J. (1990). Rasch models in latent classes: An integration of two approaches to item analysis. Applied Psychological Measurement, 14, 271-282.

Speece, D. L. (1990). Aptitude-treatment interactions: Bad rap or bad idea? The Journal of Special Education, 24, 139-149.

Vermunt, J. K., & Magidson, J. (2002a). Latent class cluster analysis. In J. A. Hagenaars& A. L. McCutcheon (Eds.), Applied latent class analysis (pp. 89-106). Cambridge, UK: Cambridge University Press.

Vermunt, J. K., &Magidson, J. (2002b). Latent class models for clustering: A comparison with K-means. Canadian Journal of Marketing Research, 20, 37-44.

Von Davier, M. (2001). WINMIRA user manual. Retrieved from http://208.76.84.140/~svfklumu/wmira/winmiramanual.pdf.

Walker, C. M., &Beretvas, S. N. (2003). Comparing multidimensional and unidimensional proficiency classifications: Multidimensional IRT as a diagnostic aid. Journal of Educational Measurement, 40, 255-275.

Webb, N. L. (2006). Identifying content for student achievement tests.InS. M.DowningT. M.Haladyna (Eds.), Handbook of test development(pp. 155-180).Mahwah, NJ: Erlbaum.

Yang, X., Shaftel, J., Glasnapp, D., &Poggio, J. (2005).Qualitative or quantitative differences?Latent class analysis of mathematical ability for special education students.The Journal of Special Education, 38, 194-207